McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 242

ANALYSIS I

Examiner: Professor R. Vermes — Date: Thursday December 13, 2007.
Associate Examiner: Professor K.Gowrisankaran — Time: 2:00 PM TO 5:00 PM

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. All questions are counted equally.
3. This is a closed book exam. No notes, crib sheets or textbooks are permitted.
4. Calculators are not permitted.
5. Use of a regular and or translation dictionary are not permitted.

This exam comprises the cover page, and 2 pages of 8 questions.

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1. Define:

   (a) Least upper bound of a set $S \subset \mathbb{R}$;

   (b) $X = (x_k)$, $x_k \in \mathbb{R}$, is a convergent sequence;

   (c) (*i*) $\lim\limits_{x \to c} f = L$, (*ii*) $\lim\limits_{x \to \infty} g = K$, (*iii*) f is continuous at c;

   (d) the real number m is the derivative of f at $x = c$.

2. (a) Prove that $\lim\limits_{n \to \infty} (\sqrt[3]{n+1} - \sqrt[3]{n}) = 0$.

   (b) Let $S = \{\sqrt[3]{n+1} - \sqrt[3]{m} : n, m \in \mathbb{N}\}$. If x and y are any real numbers with $x < y$, show that there exists an $s \in S$ such that $x < s < y$.

3. (a) Prove that $\lim\limits_{n \to \infty} n^{1/n} = 1$.

   (b) State and prove the Squeeze Theorem for sequences.

   (c) Find $\lim\limits_{n \to \infty} (n!)^{1/n^2}$. Justify your answer.

4. (a) Let f, g be functions defined on an interval $(-c, c)$, $c > 0$, and satisfying $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} g(x) = 0$. Suppose that for all $x \in (-c, c)$ the identity

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + (a_n + f(x)) x^n = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + (b_n + g(x) x^n,$$

   holds. Prove that $a_0 = b_0, a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

   (b) If the polynomial $p(x) = \sum_{k=0}^{n} a_k x^k = 0$ for all $x \in (-c, c)$, determine the coefficients a_k. Justify your answer.

5. (a) State and prove the "Maximum-Minimum" Theorem.
   (b) Let f be continuous on $[0, \infty)$. If $f(x) \geq 0$ and $\lim_{x\to\infty} f(x) = 0$ prove that there exists a $\zeta \in [0, \infty)$ such that $f(x) \leq f(\zeta)$ for all $x \in [0, \infty)$.

6. (a) Define what is meant by the statement that a function is uniformly continuous on an interval.
   (b) Let f be continuous on $\mathbb{R}$. Suppose $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both are finite, prove that f is uniformly continuous on $\mathbb{R}$.
   (c) Let f be continuous on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{2n} (-1)^k f\left(\frac{k}{2n}\right) = 0.$$

7. (a) State and prove Rolle's Theorem.
   (b) Let f be differentiable on the interval $[a, b]$, $(-\infty < a < b < +\infty)$. Suppose that $f(a) = 0, f(b) > 0$ and $f'(b) < 0$. Prove that there exists an $c \in (a, b)$ such that $f'(c) = 0$.

8. (a) State the "Cauchy Mean Value Theorem".
   (b) Let f and g be differentiable in an interval I. If for all $x \in I, f'(x) \neq 0$ and $|f'(x)| \geq |g'(x)|$, prove that

$$|f(y) - f(x)| \geq |g(y) - g(x)|$$

   for all pairs $x, y \in I$.