McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 242

ANALYSIS 1

Examiner: Professor K. Gowrisankaran
Associate Examiner: Professor S. Drury

Date: Wednesday December 13, 2006.
Time: 9:00 AM - 12:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. This is a closed book exam. Notes or books are not permitted.
3. Calculators are not permitted.
4. Use of a regular and or translation dictionary is not permitted.

This exam comprises the cover page, and 2 pages of 6 questions.

Final Examination   Math 242   Wednesday December 13, 2006.

1. (a) Let $a > 0$ and $b_n := (\sqrt{n^2 + na} - n)$ for $n \in \mathbb{N}$. Show that (b_n) converges and find the limit of (b_n).

   (b) Let $x_1 = 1, x_{n+1} := x_n + \frac{1}{x_n^2}$. Show that $(x_n) \to \infty$.

   (c) Suppose $a_n \in \mathbb{R}$ and $a_n \to 0$. Let $b_n = \frac{a_1 + \cdots + a_n}{n}$ show that (b_n) converges to 0 [Hint: Given for $\epsilon > 0$, choose a k such that $|a_j| < \frac{\epsilon}{2}$ for all $j > k$. Write $\sum_1^n a_j = \sum_1^k a_j + \sum_{k+1}^n a_j$.]

2. Decide if the following statements are true or false. Justify your conclusion in each case.

   (a) $f(x) = \frac{\sin x}{x}, x \neq 0$ and $f(0) = 1$ is a uniformly continuous function on $\mathbb{R}$.

   (b) There exists a function $g : \mathbb{R} \to \mathbb{R}$, continuous at 0 and discontinuous at 1 and verifies $g(x + y) = g(x)\, g(y)$ for all $x, y \in \mathbb{R}$.

   (c) (x_n) is a bounded sequence of real numbers and $y_n = \max(x_1, \cdots, x_n)$, then (y_n) converges to $\sup(x_n)$.

3. (a) (i) State the Bolzano-Weierstrass theorem.
   (ii) Suppose $x_n \in \mathbb{R}$. Show that either there exists a subsequence of x_n that converges to a real number or there exists a subsequence that tends to $+\infty$ or $-\infty$.

   (b) (i) Define the derivative of $c \in I$ of a function defined on an interval I (assume c is not an end point)

   (ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show $g(x) := |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

4. (a) State the Min-Max Theorem and Bolzano's Intermediate Value Theorem.

   (b) Suppose f is a continuous function on $[0, 1]$ such that $f(0) < f(1)$ and suppose further that f does not take on any of its values more than once. Show that f is strictly increasing.

5. (a) State Rolle's Theorem.

   (b) Let $q(x) = x^n + ax + b$ where $a, b \in \mathbb{R}$. Prove that q has at the most (i) two distinct real roots if n is even and (ii) three distinct real roots if n is odd.

6. (a) State the Mean Value Theorem.

   (b) Let f be continuous on $|x - c| < \alpha$ and differentiable on $0 < |x - c| < \alpha$. Suppose that $\lim_{x \to c} f'(x)$ exists and equals l. Prove that f is differentiable at c and $f'(c) = l$.