McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 242-001

ANALYSIS 1

Examiner: Professor K. GowriSankran
Associate Examiner: Professor S. Drury

Date: Monday December 12, 2005
Time: 9:00 am - 12:00 pm

INSTRUCTIONS

(a) Answer questions in the exam booklets provided.

(b) All questions carry equal weight.

(c) This is a closed book exam. No computers, notes or text books are permitted.

(d) Calculators are not permitted.

(e) Use of a regular and or translation dictionary is not permitted.

This exam comprises of the cover page, 2 pages of 6 questions.

EXAM IS PRINTED DOUBLE-SIDED

MATH 242 ANALYSIS 1
MCGILL UNIVERSITY
FINAL EXAMINATION
This is a closed book exam.
No Computers, notes or books allowed.
All questions count equally

Monday December 12, 2005. Time: 9:00 AM - 12:00 PM

1. (a) Define "$x_n \in \mathbb{R}$ and (x_n) converges to $a \in \mathbb{R}$".

   (b) Let $x_n = \sqrt{n+1} - \sqrt{n}$. Prove that (x_n) converges and also the sequence $y_n = \sqrt{n}\, x_n$ converges. Find the limit in each of the above two cases.

2. (a) Define what is meant by a Cauchy sequence.

   (b) Prove that every Cauchy sequence is bounded.

   (c) Give an example of a bounded sequence which is not Cauchy.

   (d) Suppose $x_n > 0$ for each $n \in \mathbb{N}$. Suppose also that $(x_n^{1/n})$ converges to $L < 1$. Show that $x_n \to 0$.

3. (a) State the Bolzano-Weierstrass Theorem.

   (b) Suppose $f : (0,1) \to \mathbb{R}$ is a bounded function such that $\lim_{x \to 0} f(x)$ DOES NOT EXIST. Show that there are two sequences $(x_n), (y_n)$ in (0,1) such that $\lim x_n = 0 = \lim y_n$ such that $\lim f(x_n)$ and $\lim f(y_n)$ exist but are not equal.

4. (a) State the Intermediate Value Theorem.

   (b) Suppose $f : [a, b] \to [a, b]$ is continuous. Prove that there exists at least one $x \in [a, b]$ such that $f(x) = x$.

5. (a) Define what is meant by "$f : A \to \mathbb{R}$ is uniformly continuous".

   (b) Suppose $f : A \to \mathbb{R}$ is uniformly continuous. Prove that if x_n is a Cauchy sequence in A then $(f(x_n))$ is a Cauchy sequence.

   (c) Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous. Prove that f is bounded on (a, b).

6. (a) Define what is meant by the derivative of a function f at a point $c \in I$, where f is defined on the interval I.

   (b) State the Mean Value Theorem.

   (c) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable such that the derivative f' is continuous on $[a, b]$. Show that for every $\epsilon > 0$, there exists a $\delta < 0$ such that for all x, y in $[a, b]$ with $0 <| x - y |< \delta$ we have

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon.$$