McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 242

COMPLEX VARIABLES AND TRANSFORMS

Examiner: Professor R.Vermes
Associate Examiner: Professor S. Drury

Date: Monday December 20, 2004
Time: 2:00 P.M – 5:00 P.M

INSTRUCTIONS

1. Please answer all 7 questions.
2. Calculators are not permitted.
3. Please answer in exam booklets provided.
4. This is a closed book exam.
5. No notes or text books are allowed.
6. Dictionaries are not permitted.
7. This exam consists of the cover page and 2 pages of 7 questions.

Answer questions 1-7.

1. Define:

   (a) Least upper bound of a bounded subset of $\mathbb{R}$;

   (b) $(x_k), x_k \in \mathbb{R}$, is a convergent sequence;

   (c) Λ is the upper limit (limit superior) of a bounded real sequence (x_k);

   (d) f is continuous at c;

   (e) f is uniformly continuous on a set $S \subset \mathbb{R}$;

   (f) A real number m is the derivative of f at c.

2. (a) State the Least Upper Bound Axiom.

   (b) Let A and B non-empty bounded sets of positive numbers. Define a set C by

   $$C = \{ab : a \in A, b \in B\}.$$

   Prove that $\sup C = \sup A \sup B$.

3. (a) State and prove the "squeeze" theorem for sequences.

   (b) Find: $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n(n+1)}} - 1 \right).$

4. (a) State the Cauchy convergence criterion for sequences.

   (b) Let (x_n) be a sequence and let r be a real number that satisfies $0 < r < 1$. Suppose that $|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|$ for all $n > 1$. Prove that the sequence (x_n) converges.

5. (a) Let f be continuous on the interval $[a, b], (-\infty < a < b < \infty)$, prove that f is bounded on $[a, b]$.

   (b) Let f be uniformly continuous function on the open interval (a, b), $(-\infty < a < b < \infty)$. Prove that f is bounded on (a, b). Show that the conclusion is false on the interval (a, ∞).

6. (a) State and prove the Intermediate Value Theorem.

   (b) A function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there exits a positive number p such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function, prove that the equation $f(x) - x = 0$ has a solution.

7. (a) State Rolle's Theorem.

   (b) State and prove The Mean Value Theorem of Differential Calculus.

   (c) Let f be a continuous function on $(-1, 1)$ and differentiable on $\{x : 0 <| x |< 1\}$. If $\lim_{x \to 0} f'(x)$ exists, show that f is differentiable also at $x = 0$.

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FINAL EXAM

MATH 248

ADVANCED CALCULUS 1

Examiner: Professor R. Rigelhof
Associate Examiner: Professor I. Klemes

Date: Tuesday December 21, 2004
Time: 9:00 a.m -12:00 p.m

INSTRUCTIONS

1. Please answer all 12 questions.
2. Please answer in the exam booklets provided.
3. This is a closed book exam.
4. No books or notes are permitted
5. No calculators are allowed.
6. No Dictionaries are allowed.
7. This exam consists of the cover page and 2 pages of 12 questions .

Examiner: R. Rigelhof
Associate Examiner: I. Klemes

All questions are of equal value. Do question 1 **OR** 2 and nine of the remaining questions.

1. Show that the equations

$$xy^2 + zu + v^2 = 3$$
$$x^3z + 2y - uv = 2$$
$$xu + yv - xyz = 1$$

can be solved for x, y and z in terms of u and v near the point where

$$(x, y, z, u, v) = (1, 1, 1, 1, 1)$$

and evaluate $\frac{\partial y}{\partial u}$ at $(u, v) = (1, 1)$.

2. Show that the equations $x = u^2 + v^3$ and $y = uv - v^3$ can be solved for u and v in terms of x and y, near the point where $u = v = 1$ and evaluate:

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial(u, v)}{\partial(x, y)}$ at the point $(u, v) = (1, 1)$

3. If $F(x, y, z) = 0$ determines z as a function of x and y, calculate $\frac{\partial^2 z}{\partial x^2}$ in terms of the partial derivatives of F.

4. Change the order of integration and evaluate the integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3} dy$.

5. Compute the volume of the intersection of two cylinders of radius a, one with axis on the x-axis, the other with axis on the y-axis.

6. Compute the surface area of that part of the sphere $x^2 + y^2 + z^2 = 4a^2$ which is inside the cylinder $x^2 + y^2 = 2ax$.

7. Use spherical coordinates to set up the integral to calculate the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cone $z^2 = x^2 + y^2$.

8. Show that for a smooth scalar function ϕ and a smooth vector function $\vec{F}$, the following vector identity is valid:

$$\nabla\times\phi\vec{F} = (\nabla\phi)\bullet\vec{F}+\phi(\nabla\bullet\vec{F})$$

9. Let C be a positively oriented, simple closed curve in the xy-plane, bounding a region R and not passing through the origin. Show that

$$\oint_C \frac{xdy - ydx}{x^2+y^2} = 2\pi$$

if the origin is inside the region R, and 0 if the origin is outside R.

10. Show that the flux of the field $\vec{F} = m\vec{r}/|r|^3$ through any sphere centred at (0,0,0) is 4π, and hence that the flux through any smooth oriented surface with outward directed normal which is the boundary of a regular domain containing the point (0,0,0) is 4π.

11. Let C_1 be the straight line joining (-1,0,0) to (1,0,0), and let C_2 be the semicircle $x^2+y^2 = 1, z = 0, y\ge 0$. Let S be a smooth surface joining C_1 to C_2 having upward normal, and let

$$\vec{F}(x,y,z) = (\alpha x^2-z)\vec{i}+(xy+y^3+z)\vec{j}+\beta y^2(z+1)\vec{k}$$

Find the values of α and β for which $I = \iint_S \vec{F}\bullet d\vec{S}$ is independent the choice of S, and find the value of I for these values of α and β.

12. Let C be the curve $(x-1)^2+4y^2 = 16, 2x+y+z = 3$, oriented counter clockwise when viewed from high on the z-axis. Let

$$\vec{F}(x,y,z) = (z^2+y^2+\sin(x^2))\vec{i}+(2xy+z)\vec{j}+(xz+2yz)\vec{k}$$

Use Stokes' Theorem to evaluate $\oint_C \vec{F}(x,y,z)\bullet d\vec{r}$.