Final Examination MATH 242 December 16, 2003

Answer questions 1-7, and question 8 is for extra marks.

1. (a) Define:
    i. Least upper bound of $S \subset \mathbb{R}$;
    ii. $(x_k), x_k \in \mathbb{R}$, is a convergent sequence;
    iii. Cauchy sequence;
    iv. f is uniformly continuous on a set $S \subseteq \mathbb{R}$;
    v. A real number m is the derivative of f at c.

   (b) i. Define: $\lim_{x \to \infty} f(x) = L$;
    ii. Let $f : [0, \infty) \to \mathbb{R}$ be periodic (ie., $\exists p > 0$ such that $f(x + p) = f(x)$ for all $x \in [0, \infty)$. Suppose $\lim_{x \to \infty} f(x) = 0$, prove that $f(x) = 0$ for all $x \in [0, \infty)$.

2. (a) State the Least Upper Bound Axiom (Completeness Axiom).

   (b) Let S be any subset of $\mathbb{R}$. If for $x \in \mathbb{R}$ $f(x) = \text{dis}(x, S) = \inf\{|x - s| : s \in S\}$, prove that $|f(x) - f(y)| \leq |x - y|$.

   (c) Is the function f in (b) uniformly continuous on $\mathbb{R}$? Justify your answer.

3. (a) i. Let $a > 0$, prove that $\lim_{n \to \infty} \sqrt[n]{a} = 1$;
    ii. Suppose $a \geq 1$, show that $\lim_{n \to \infty} n(\sqrt[n]{a} - 1)^2 = 0$.

   (b) Suppose that (x_n) is a convergent sequence and (y_n) is a sequence such that for any $\varepsilon > 0$ we can find an $n_0 \in \mathbb{N}$ such that $|x_n - y_n| < \varepsilon$ if $n \geq n_0$. Does it follow that (y_n) is a convergent sequence? Justify your answer.

4. (a) Let f be a continuous function defined on the closed and bounded interval $[a, b]$. If $f(a) < \gamma < f(b)$, prove that there exists a number $c \in (a, b)$ such that $f(c) = \gamma$.

   (b) Let f be continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists x_1 and x_2 in $[0, 2]$ such that $x_2 = 1 + x_1$ and $f(x_2) = f(x_1)$

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5. (a) Let f be defined on the open interval (a, c). Suppose that for some $b \in (a, c)$ f is uniformly continuous on $(a, b]$ and on $[b, c)$, prove that f is uniformly continuous on (a, c).

   (b) Let $f : [a, \infty) \to \mathbb{R}$ be continuous and $\lim_{x\to\infty} f(x) = L$. Prove that f is uniformly continuous on $[a, \infty)$.

6. (a) State and prove Rolle's Theorem.

   (b) Show that the polynomial $p(x) = x^n + ax + b, (n \geq 2)$, ($a, b$ arbitrary real numbers), has at the most two distinct zeros for n even and at most three distinct zeros for n odd.

   (c) Prove that the equation $x^5 - 4x + 2 = 0$ has three real solutions.

7. (a) State and prove the mean value theorem of Lagrange.

   (b) State the mean value theorem of Cauchy.

Question 8 is for extra marks, work on it only if you have solved Questions 1-7.

8. (a) Let f be continuous on $a \leq x < b$ and differentiable on (a, b). If $f(a) = 0$ and $f(x) > 0$ for $x \in (a, b)$, prove that there can not be a constant M such that $0 \leq \dfrac{f'(x)}{f(x)} \leq M$ for $a < x < b$.

   (b) Let $0 < a < b$. If f is continuous on $[a, b]$ and differentiable on (a, b), show that there exists a $\zeta \in (a, b)$ such that

$$\frac{bf(a) - af(b)}{b - a} = f(\zeta) - \zeta f'(\zeta).$$

   [Hint: Let $F(x) = \dfrac{f(x)}{x}$ and choose a $G(x)$.]