1. (i) (4 marks) State the Triangle Inequality for scalars.
   (ii) (4 marks) State the Cauchy–Schwarz Inequality.
   (iii) (8 marks) Prove by induction on n that

$$\left|\sum_{j=1}^{n} a_j\right| \leq \sum_{j=1}^{n} |a_j|$$

   whenever $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are real numbers.
   (iv) (4 marks) Show that

$$\left|\sum_{j=1}^{n} a_j\right| \leq \sqrt{n}\left\{\sum_{j=1}^{n} a_j^2\right\}^{\frac{1}{2}}$$

   whenever $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are real numbers.

2. (i) (4 marks) State the Bolzano Intermediate Value Theorem.
   (ii) (4 marks) State the Mean-Value Theorem.

   Let $f(x) = x^7 + 2x^3 - 21$.

   (iii) (4 marks) Show that $f'(x) > 0$ if $x \neq 0$ and that $f''(x) > 0$ if $x > 0$.
   (iv) (4 marks) Show that the equation $f(x) = 0$ has one and only one real solution $x = \xi$ and that $1 < \xi < 2$.
   (v) (4 marks) Let $x > \xi$. Show that $x - \dfrac{f(x)}{f'(x)} > \xi$ by applying the Mean-Value Theorem to f on the interval $[\xi, x]$.

3. (i) (4 marks) State a theorem about the convergence of monotone sequences of real numbers.
   (ii) (4 marks) State the Nested Intervals Theorem. Be sure to include the addendum to the Nested Intervals Theorem.

   In this question, you may use all the parts of the previous question without proof. Let $f(x) = x^7 + 2x^3 - 21$. Let $x_1 = 2$ and $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ for $n = 2, 3, \ldots$

   (iii) (4 marks) Show that (x_n) is a decreasing sequence of real numbers and that it is bounded below by ξ. What can you deduce?
   (iv) (8 marks) Show that $x_n \longrightarrow \xi$ as $n \to \infty$.

4. (i) (4 marks) What is meant by a Cauchy sequence of real numbers?

Let c be any real number with $0 \le c \le \frac{2}{3}$. Let $f(x) = \dfrac{x}{2+x^2} + c$. Let $x_1 = 0$ and define x_n inductively by $x_n = f(x_{n-1})$ for $n = 2, 3, \ldots$.

(ii) (4 marks) By finding suitable upper and lower bounds for f on $[0,1]$, show that $0 \le x_n \le 1$ for all $n \in \mathbb{N}$.
(iii) (4 marks) Find an upper bound for $\{|f'(x)|; 0 \le x \le 1\}$.
(iv) (4 marks) Show that (x_n) is a Cauchy sequence of numbers in $[0,1]$.
(v) (4 marks) What can you deduce for the sequence (x_n)?

5. (i) (4 marks) Define the concept of a subsequence of a sequence of real numbers
(ii) (4 marks) State the Bolzano-Weierstrass Theorem.

Suppose that a bounded sequence of real numbers $(x_n)_{n=1}^{\infty}$ does not converge to x.

(iii) (6 marks) Show that there exists a strictly positive number ϵ and a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $|x - x_{n_k}| \ge \epsilon$ for all $k \in \mathbb{N}$.
(iv) (6 marks) Deduce that $(x_n)_{n=1}^{\infty}$ possesses a subsequence which converges to a point different from x.

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.
(i) (4 marks) What does it mean for f to be continuous?
(ii) (4 marks) What does it mean for f to be uniformly continuous?

For each of the following functions, determine whether the function is uniformly continuous. Justify your answer.

(iii) (6 marks) $f(x) = x^3$ defined on $\mathbb{R}$.
(iv) (6 marks) $f(x) = \dfrac{x}{1+x^2}$ defined on $\mathbb{R}$.

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-242A

Analysis I

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Monday, December 11, 2000
Time: 9: 00 am. – 12: 00 noon

INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.