1. (a) Define what is meant by a bounded sequence, a convergent sequence, a monotone sequence. Prove that a convergent sequence is bounded. Give an example to show that a bounded sequence need not be convergent.

   (b) If (a_n) is a convergent sequence, (b_n) a bounded sequence and $\lim(a_n) = 0$, show that $(a_n b_n)$ is convergent and $\lim(a_n b_n) = 0$.

2. (a) If S is a bounded nonempty set in $\mathbb{R}$, show that the set $S' = \{x : -x \in S\}$ is bounded, and determine its bounds in terms of the bounds of S.

   (b) Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A,\ b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$.

3. (a) If $a_n = \dfrac{1}{n+1} + \ldots + \dfrac{1}{2n}$, for all $n \in \mathbb{N}$, show that $\dfrac{1}{2} \leq a_n \leq 1$ for all n; show further that (a_n) converges.

   (b) If $a_n = \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$, show that (a_n) and $(\sqrt{n}a_n)$ both converge.

4. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that f attains an absolute maximum on $[a, b]$.

   (b) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and let

   $$\lim_{x \to a+} f(x) = \lim_{x \to b-} f(x) = 0.$$

   If $f\left(\dfrac{a+b}{2}\right) > 0$ prove that f attains an absolute maximum at some point $c \in (a, b)$.

5. Define what is meant by the statement that a function f is *uniformly continuous* on an interval. *State* a theorem on the uniform continuity of a continuous function on a bounded closed interval. Show that a function f can be uniformly continuous on every bounded interval and yet not be uniformly continuous on $\mathbb{R}$. Show however that if in addition $\lim\limits_{x \to +\infty} f(x)$, $\lim\limits_{x \to -\infty} f(x)$ both exist and are finite, then f *is* uniformly continuous on $\mathbb{R}$.

6. State the Mean-Value Theorem.
   Let f be a differentiable function in $\mathbb{R}$ with $|f'(x)| \leq \dfrac{1}{2}$ for all $x \in \mathbb{R}$. Prove that for any $x, y \in \mathbb{R}$

   $$|f(x) - f(y)| \leq \frac{1}{2}|x - y| \ .$$

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-242A

REAL ANALYSIS I

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury

Date: Tuesday, December 10, 1996
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

No Calculators, Notes or Books Permitted
All questions carry equal marks

This exam comprises the cover and 1 page of questions.