MATHEMATICS 189–236B

PART I

DO ANY THREE QUESTIONS FROM AMONG THE FOLLOWING SIX

1. Let F_q be a finite field with q elements.

   (a) Show that there is a prime p such that $q = p^k$ for some positive integer k.

   (b) Show that a vector space of dimension n over F_q has q^n elements.

   (c) Find a formula for the number of ordered bases for a vector space V of dimension n over F_q.

   (d) How many $n \times n$ matrices are there with entries from F_q?

   (e) How many invertible $n \times n$ matrices are there over with entries from F_q?

2. Consider the matrix A whose entries are from the field F_2 of two elements (recall that in this field $1 + 1 = 0$). The subspace U of $\mathcal{R}^5$ is spanned by the first three columns of A and the subspace V by the last three columns.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

   (a) Find an invertible matrix B such that BA is in reduced row echelon form.

   (b) Find all dependence relations on the rows of A.

   (c) Find all dependence relations on the columns of A.

   (d) Find bases for U, V, $U + V$, and $U \cap V$.

3. Test each of the following sets to see if it is a subspace. In each case your answer must be justified–if yes, then why; if no then why not.

   (a) $\{(x, y, z)^t | x + 3y - z = 5\} \subset \mathcal{R}^3$.

   (b) $\{\mathbf{v}|$ there is $\mathbf{w}$ with $S\mathbf{v} = T\mathbf{w}\} \subset \mathcal{R}^n$ where S and T are two matrices of suitable sizes.

   (c) The set of sequences of complex numbers a_n with $\sum_{n=0}^{\infty} a_n^2 < \infty$ considered as a subset of the vector space of all sequences of complex numbers.

   (d) i. Over the real numbers, the set of integrable functions f(x) which vanish at $x = 0$ considered as a subset of the space of all real valued functions.

   ii. as above, but the functions such that $f(0) = 1$.

4. Let $\mathcal{B}$ be a linearly independent subset of the vector space V whose field of scalars is $\mathbf{F}$ and $\mathbf{v} \in V$ a vector in V.

   (a) Prove that the following two statements are equivalent

   i. $\mathcal{B} \cup \{\mathbf{v}\}$ is a linearly independent subset of V.

   ii. $\mathbf{v} \notin \text{SPAN}\mathcal{B}$.

6. (a) State the definition given in class of a determinant function which assigns an $n \times n$ matrix with entries from the field $\mathbf{F}$ a scalar in the field $\mathbf{F}$ which is related to the behaviour of the determinant under elementary operations.

   (b) State the formula for calculating the determinant (this involves the signs of permutations).

   (c) Derive the cofactor expansion along the first row using the formula of part (b) of this question. Recall that $C_{ij} = (-1)^{i+j} M_{ij}$ whereby M_{ij} is the minor determinant of the ij position.

PART II

DO ANY THREE OF THE FOLLOWING FIVE QUESTIONS

1. For the matrix M given below, compute and factor the characteristic polynomial (one of its roots is -2), then using the idea of orthogonal idempotents and the Cayley–Hamilton theorem, find a basis of $\mathbf{R}^3$ which puts the matrix into Jordan canonical form (the block diagonal form discussed in class and on two assignments).

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{pmatrix}$$

2. Let V be the vector space of polynomials of degree at most two over the real numbers $\mathcal{R}$ equipped with the inner product

$$< f, g >= \int_{-1}^{1} f(t)g(t)dt$$

   (a) Apply the Gram–Schmidt process to the basis $\{1, x, x^2 + x\}$ to obtain an orthonormal basis.

   (b) With respect to the standard basis $\{1, x, x^2\}$ find the **second** column of the matrix of the adjoint of the derivative operator D.

3. (a) Find a rotation matrix P which diagonalises the quadratic form

$$q(x, y, z) = x^2 + y^2 + 4z^2 - 2xy + 4xz - 4yz$$

   Note: zero is an eigenvalue of the associated symmetric matrix.

   (b) Prove that $\lambda = 1$ is an eigenvalue of the ROTATION MATRIX P. What is the geometric interpretation of the line of action of the corresponding eigenvalue?

4. Let T be a self–adjoint linear transformation on the finite dimensional inner product space V over $\mathcal{C}$ the field of complex numbers and U a T–invariant subspace.

   Show that $U^{\perp} = \{\mathbf{w} \in V | < \mathbf{w}, \mathbf{u} >= 0 \text{ for all } \mathbf{u} \in U\}$ is also a T-invariant subspace of V

5. Let T be a self–adjoint linear transformation on the finite dimensional inner product space V over $\mathcal{C}$ the complex numbers.

   Show that there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V consisting entirely of eigenvectors of T.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-236B

LINEAR ALGEBRA I

Examiner: Professor W. Jonsson
Associate Examiner: Professor J. Loveys

Date: Wednesday, April 30, 1997
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

There are three parts to this exam.
Please read the instructions at the top of each section carefully.

This exam comprises the cover and 2 pages of questions.