Faculty of Science

FINAL EXAMINATION

MATH 223

Linear algebra

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Date: April 30, 2007
Time: 9 A.M. - 12 P.M.

INSTRUCTIONS

There are 8 problems, altogether worth 90 marks.
Answer all questions in examination booklets.
Show all necessary steps and details in your work.
Notes and textbooks are not permitted.
Neither regular nor translation dictionaries are allowed.
You can use only non-graphing, non-programmable calculators.

This exam comprises the cover and 2 pages of questions.

Problem 1.[**10 MARKS**] Find bases for the row, column and null spaces of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 8 & 11 & 19 & 0 & 11 \end{bmatrix}$$

Problem 2. Let V be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let $W = V^{\perp}$ be the orthogonal complement of V.

(a) [**4 MARKS**] Find an orthogonal basis for W

(b) [**6 MARKS**] Let T be the orthogonal projection onto W. Find the matrix of T using the basis in part (a).

Problem 3. Let P_2 be the space of polynomials of degree at most 2. Suppose that

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is the matrix for a linear transformation $T : P_2 \to P_2$ with respect to the basis $\mathfrak{B} = \{p_1(x), p_2(x), p_3(x)\}$ of P_2, where $p_1(x) = 1 + x$, $p_2(x) = 1 + x^2$, and $p_3(x) = x + x^2$

(a) [**4 MARKS**] Find $[T(p_1)]_{\mathfrak{B}}$, $[T(p_2)]_{\mathfrak{B}}$, and $[T(p_3)]_{\mathfrak{B}}$.

(b) [**4 MARKS**] Find $T(p_1)$, $T(p_2)$, and $T(p_3)$.

(c) [**6 MARKS**] Find the standard matrix of T with respect to the basis $\mathfrak{A} = \{1, x, x^2\}$.

Problem 4.[**10 MARKS**] Find the least-squares solution $\vec{x}^*$ of the system

$$A\vec{x} = \vec{b}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Determine the error $\|\vec{b} - A\vec{x}^*\|$.

Problem 5. Let $V = C[0,1]$ be the vector space of continuous functions on the interval $[0,1]$. For f, g in V let

$$\langle f, g\rangle = \int_0^1 (1-t)f(t)g(t)dt$$

(a) **[4 MARKS]** Show that $\langle .,. \rangle$ defines an inner product on V.

(b) **[6 MARKS]** Let W be the subspace of V spanned by $f_1 = 1$ and $f_2 = t$ for t in $[0,1]$. Compute the matrix $A = [a_{ij}]$, $1 \leq i, j \leq 2$, where $a_{ij} = \langle f_i, f_j \rangle$.

(c) **[6 MARKS]** Find an orthonormal basis of W.

Problem 6.[8 MARKS] Compute the determinant of the matrix

$$A = \begin{bmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c \end{bmatrix}$$

where a, b, c, x and y are real numbers.

Problem 7. Let

$$A = \begin{bmatrix} i & 0 \\ 2 & -1 \end{bmatrix}$$

where i stands for the imaginary unit, i.e, $i^2 = -1$.

(a) **[4 MARKS]** Determine all eigenvalues of the matrix A.

(b) **[4 MARKS]** Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

(c) **[6 MARKS]** Give a general formula for A^n, where n is a positive integer.

Problem 8. Suppose that A is an orthogonal matrix with real entries. Show that

(a) **[4 MARKS]** A^{-1} is an orthogonal matrix.

(b) **[4 MARKS]** $det(A) = \pm 1$.