McGILL UNIVERSITY
FACULTY OF SCIENCE
FINAL EXAMINATION

MATHEMATICS 223 2006 01 — LINEAR ALGEBRA

EXAMINER: Professor W. G. Brown
ASSOCIATE EXAMINER: Prof. J. Loveys

DATE: Thursday, April 20th, 2006
TIME: 14:00 – 17:00 hours

FAMILY NAME:
GIVEN NAMES:

STUDENT NUMBER:

INSTRUCTIONS

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in. You may do rough work for this paper anywhere in the booklet.
3. Calculators are not permitted. This is a closed book exam. DICTIONARIES ARE ALLOWED.
4. This examination booklet consists of this cover, Pages 1 through 7 containing questions; and Pages 8, 9, 10, and 11, which are blank. *Your neighbour's version of this examination may be different from yours.*
5. All questions on this paper require that you show all your work. Their solutions are to be written in the space provided on the page where the question is printed. When that space is exhausted, you may write *on the facing page*. Any solution may be continued on the last pages, or the back cover of the booklet, but you must indicate any continuation clearly on the page where the question is printed!
6. **You are responsible for verifying the correctness of your calculations wherever that is possible.**
7. You are expected to simplify your answers wherever possible.

   You are advised to spend the first few minutes scanning the problems. (Please inform the invigilator if you find that your booklet is defective.)
8. A TOTAL OF 70 MARKS ARE AVAILABLE ON THIS EXAMINATION.

PLEASE DO NOT WRITE INSIDE THIS BOX

1(a) /2	1(b) /4	1(c) /2	1(d) /2	2(a) /2	2(b) /3	2(c) /5	3(a) /6	3(b) /6
4(a) /3	4(b) /6	4(c) /1	5(a) /4	5(b) /3	5(c) /3	6 /10	7 /10	
								TOTAL /70

1. Let $\mathbf{F}(\mathbb{R})$ denote the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

   (a) [2 MARKS] Define precisely what is meant by the statement:

   The functions $f_1, f_2, \ldots, f_n \in \mathbf{F}(\mathbb{R})$ are linearly independent in $\mathbf{F}(\mathbb{R})$.

   (b) [4 MARKS] Show carefully that the three functions $t\,,\quad t^2\,,\quad t^3$ are linearly independent in $\mathbf{F}(\mathbb{R})$.

   (c) [2 MARKS] Show that the four functions $t\,,\quad t^2\,,\quad t^3\,,\quad 0$ are not linearly independent in $\mathbf{F}(\mathbb{R})$.

   (d) [2 MARKS] Show that the following three functions are linearly dependent in $\mathbf{F}(\mathbb{R})$:

$$\sin^2(t)\,,\quad -\cos^2(t)\,,\quad 4\,.$$

2. Let $\mathbb{R}_4[t]$ denote the real vector space of polynomials of degree not exceeding 4. Define the function $F : \mathbb{R}_4[t] \to \mathbb{R}_4[t]$ as follows: for any polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \in \mathbb{R}_4[t]$, $F(p(t)) = p(-t)$; that is, the polynomial $p(t)$ is mapped on to the polynomial function whose value at a point $t \in \mathbb{R}$ is defined to be $p(-t)$.

   (a) [2 MARKS] Show that the function F is a linear transformation.

   (b) [3 MARKS] Show that the function G which maps a polynomial $p(t)$ on to the polynomial $p(t) + 1$ is *not* a linear transformation.

   (c) [5 MARKS] Assuming that the following are coordinate systems for $\mathbb{R}_4[t]$, with the vectors in the systems ordered as written,

$$\begin{aligned} B &= \{1, 2t, 3t^2, 4t^3, 5t^4\} \\ C &= \{1, t, t + t^2, t + t^2 + t^3, -1 + t^4\}, \end{aligned}$$

   find the matrix $[F]_B^C$, i.e., using C as the coordinate system for the domain, and B as the coordinate system for the target.

3. Let $\mathbb{R}_{2,2}$ be the real vector space of 2×2 real matrices. For any two such matrices, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, define

$$\begin{aligned} f(A,B) &= a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + 4a_{22}b_{22} \\ g(A,B) &= a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} \\ h(A,B) &= a_{11}b_{11} + 2a_{12}b_{21} + 3a_{21}b_{12} + 4a_{22}b_{22} \end{aligned}$$

(a) [6 MARKS] Show that f defines an inner product.

(b) [4 MARKS] Then show, *by providing a specific counterexample in each case*, that neither g nor h defines an inner product.

4. Let $C = \begin{pmatrix} 2 & 3 & -4 \\ 0 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}$.

   (a) [3 MARKS] Showing all your work, determine all the eigenvalues of C.

   (b) [6 MARKS] Determine a maximal set of linearly independent eigenvectors corresponding to each of the eigenvalues.

   (c) [1 MARK] Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

5. Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ 2 & 4 & 3 & 7 & 4 \\ -1 & -2 & -2 & -5 & -6 \\ 3 & 6 & 6 & 14 & 15 \end{pmatrix}.$$

(a) [4 MARKS] Determine a basis for the row space of A.

(b) [3 MARKS] Determine a basis for the image of the linear transformation determined by A.

(c) [3 MARKS] Determine a basis for the kernel (or null space) of A.

Note: You are responsible for verifying the correctness of your calculations.

6. [10 MARKS] Using the Gram-Schmidt Process and the dot product, determine an orthonormal set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^3$ for which

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix}\right\}.$$

Note: You are responsible for verifying the correctness of your calculations.

7. [10 MARKS] Let the matrix of the linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ referred to the standard coordinate system S be $[F]_S^S = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{pmatrix}$.

Showing all your work, determine the matrix $\left[F^{-1}\right]_B^B$ that represents F^{-1} referred to the coordinate system

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

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