McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 223

LINEAR ALGEBRA

Examiner: Professor Olga Kharlampovich
Associate Examiner: Professor Jim Loveys

Date: Tuesday April 26, 2005
Time: 9:00AM - 12:00PM

INSTRUCTIONS

1. Please answer all questions in exam booklets provided.
2. This is a closed book exam.
3. Simple Pocket Calculators that have no scientific functions are permitted only.
4. Regular or Translation dictionaries are not permitted.

This exam comprises the cover page, and 2 pages of 8 questions.

MATH 223 Final exam
Tuesday April 26, 2005 9:00 AM - 12:00 PM
Solve all problems.

1. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal.

(c) Find a matrix B such that $B^2 = A$.

(d) Find $f(A)$, where $f(t) = t^4 - 3t^3 - 6t^2 + 7t + 3$.

2. Let

$$A = \begin{bmatrix} 1 & 6 & a \\ -5 & -6a & -25 \\ a & 30 & 25 \end{bmatrix}.$$

Find the rank of A for all different values of a.

3. Find the Fourier coefficient c and the projection cw of $v = (3 + 4i,\ 2 - 3i)$ along $w = (5 + i,\ 2i)$ in $\mathbb{C}^2$.

4. Prove that if $\{u_1, \ldots, u_r\}$ is an orthogonal set of vectors, then

$$||u_1 + \cdots + u_r||^2 = ||u_1||^2 + \cdots + ||u_r||^2.$$

5. Let $V = C[-\pi, \pi]$ be the vector space of continuous functions on the interval $[-\pi, \pi]$ with inner product defined by $< f, g >= \int_{-\pi}^{\pi} f(t)g(t)dt$. The following is an orthogonal set in V

$$\{1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x, \ldots\}.$$

Find the Fourier coefficients of $f(x) = x$, namely the numbers a_0, b_k, c_k (note that b_k and c_k may depend on k) such that

$$f(x) = a_0 1 + b_1 \sin x + c_1 \cos x + b_2 \sin 2x + c_2 \cos 2x + \ldots.$$

6. Let $F(x, y) = (3x + 4y, 2x - 5y)$. Find the matrix of F in the standard basis and <u>also</u> in the basis $S = \{u_1, u_2\}$, where $u_1 = (1, 2),\ u_2 = (2, 3)$.

7. Determine whether the given set S is a subspace of the vector space $\mathbb{R}^{4\times 4}$ of 4×4 matrices. If S is a subspace of V compute the dimension of S. Explain your answer.

(a) S is the set of 4×4 matrices whose entries are all integers;

(b) S is the set of 4×4 matrices whose entries are all greater than or equal to 0;

(c) S is the set of 4×4 matrices with trace 0;

(d) S is the set of all upper triangular 4×4 matrices;

(e) S is the set of 4×4 matrices such that the vector $\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$ is in the null space of A;

(f) S is the set of 4×4 matrices with non-zero determinant.

8. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & -2 & 3 & -1 \end{bmatrix}$

Find orthonormal bases of:

(a) the null space of A,
(b) the row space of A,
(c) the image of the linear mapping given by A.