1. Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & -1 \\ 5 & 1 & 6 & 6 & -2 \\ 13 & 8 & 21 & 12 & -7 \end{pmatrix}$$
and $\vec{b} = (1, 5, 13)^t$. Find a basis for the row space, column space and null space of A. Give the dimension of each space. What is the rank of A? Solve $A\vec{v} = \vec{b}$. Is $(-7, -8, -3, -12, 7)$ in the row space? Is $(-1, -1, 2, 0, 3)^t$ in the null space?

2. $W = span\{(1, 1, 0, 1)^t, (1, 0, 1, 1)^t\}$. Find orthogonal bases for W and $W^\perp$; the inner product is the usual dot product. Check directly that the collection of all the vectors in the two bases is independent; this will verify that $W \oplus W^\perp = \mathcal{R}^4$. ($\mathcal{R}$ is the field of real numbers.) Express $(4, 3, 2, 1)^t$ as a sum $\vec{w} + \vec{u}$, where $\vec{w} \in W$ and $\vec{u} \in W^\perp$.

3. $y_1(t)$ and $y_2(t)$ are functions in the real variable t. Solve the system of differential equations
$$\begin{aligned} y_1' &= 5y_1 - 5y_2 \\ y_2' &= 5y_1 + 5y_2 \end{aligned}$$
first over $\mathcal{C}$ (the complex numbers) and then over $\mathcal{R}$. If $y_1(0) = y_2(0) = 3$, give the particular solution.

4. For each of the following subsets of $V = P_3(t)$ (the collection of polynomials of degree ≤ 3), state which are subspaces of $P_3(t)$; justify.
   (a) The set $P_2(t)$ of polynomials of degree ≤ 2.
   (b) $\{f \in V : f'(6) = 0\}$.
   (c) $\{f \in V : f'(7) = 1\}$.

5. T is the function from $\mathcal{C}^2$ to itself defined by $T[(z_1, z_2)^t] = (z_1 + z_2, 6z_1)^t$. Verify that T is a linear transformation. Give the matrix of T with respect to the standard basis, and also with respect to the basis $\{(i, 0)^t, (0, 1 + i)^t\}$. Give the change-of-basis matrix.

6. Define the operator $T : P_3(t) \to P_3(t)$ by the formula $T(f) = f + \dfrac{df}{dt}$.

   (a) Prove that T is linear.

   (b) Let $\mathcal{F} = (t+t^3, -2t+t^3, 1+7t-5t^2, 1-8t^3)$. $\mathcal{F}$ is a basis of $P_3(t)$ (do not check it). Write down the product of specific matrices and inverses of such that gives $[T]_{\mathcal{F}}$, the matrix of T with respect to the basis $\mathcal{F}$. Do not calculate $[T]_{\mathcal{F}}$.

7. Verify that, for the space V of continuous real-valued functions on $[0, \frac{\pi}{2}]$, the function $< f|g >= \int_0^{\frac{\pi}{2}} sin(t) f(t) g(t) dt$ is an inner product. Let $W = span\{\sin(t), \cos(t)\}$. Find an orthogonal basis for the subspace W. [Hint: the easiest way to integrate $\sin^2(t)\cos(t)$ is to use the substitution $u = \sin t$, $du = \cos t dt$. $\sin^3 t = \sin t - \cos^2(t)\sin t$, so make another substitution.] Write the formula for, but do not calculate, an orthonormal basis of W.

8. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & -2 \end{pmatrix}.$$

   Use this to find A^6.

9. Describe the shape of the surfaces in $\mathcal{R}^3$ whose equations are:

   (a) $5x^2 - 4xy + 2y^2 + 6z^2 = 1$;

   (b) $x^2 - 4xy + 4y^2 + 2xz - 4yz + z^2 = 1$;

   (c) $-x^2 + 4xz + 2y^2 + 2z^2 = 1$.

10. V is the vector space of 2×2 matrices over $\mathcal{R}$ and $T : V \longrightarrow V$ is the operator defined by $T(A) = \frac{1}{2}(A + A^t)$. Show that T is a projection. What is the rank of T? What is $Im(T)$? What is $Ker(T)$?

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-223A

LINEAR ALGEBRA

Examiner: Professor M. Makkai
Associate Examiner: Professor J. Loveys

Date: Monday, December 13, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are not permitted.

This exam comprises the cover and two pages of questions.