McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 222

Calculus 3

Examiner: Gil Alon
Associate Examiner: Professor Wilbur Jonsson

Date: Wednesday April 27, 2005
Time: 2:00PM - 5:00PM

INSTRUCTIONS

Please answer all questions in exam booklets provided.
This is a closed book exam.
Calculators are not permitted.
Regular or Translation dictionaries are not permitted.

This exam comprises the cover page, and 1 page of 8 questions.

McGill University - Dept. of Mathematics and Statistics

Math 222- Calculus 3

Final Examination - 2005 Winter term

Examiner: Dr. G. Alon

Associate examiner: Prof. W. Jonsson

Answer the following 8 questions.
You do not have to simplify numerical answers (e.g, $\sin 5 - \log 4$ is an acceptable answer).

1. (13 points) Find the radius of convergence of each of the following power series:

   (a) $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} n^2 (x-1)^n$

   (b) $\sum_{n=1}^{\infty} \frac{2^n x^{2n}}{n}$.

2. (13 points)
   (a) Find the first two nonzero terms in the Maclaurin series of $f(x) = \cos^5 x$ and find an estimate for the sum of the remaining terms of the series, for $0 \leq x \leq 0.1$.
   (b) Find the Maclaurin series of $\arctan t$ by integrating $\frac{1}{1+t^2}$. Use this expansion to find $f^{(10)}(0)$ where $f(x) = \arctan 5x^2$.

3. (13 points) Find the equations of the tangent plane and the normal vector to the given surface at the given point, for:

   (a) $(x, y, z) = (2se^t, \cos s + t, \frac{s}{t})$, at $s = t = 2$

   (b) $(x^2 + y^2 + z^2)^2 + \sqrt{yz} = 10$ at $x = y = z = 1$.

4. (12 points)
   (a) Find the arc length of the curve $r(t) = (3t^2, 4\sqrt{2}t^{1.5}, 6t)$ between $r(0)$ and $r(1)$.
   (b) Write the arc length parametrization of this curve. The answer should be a function of the parameter s.
   (c) If $P = (12, 16, 12)$ and $f(x, y, z)$ is a differentiable function defined in a neighbourhood of P and satisfying $\nabla f(P) = (1, 1, 2)$, find $\frac{d}{dt} f(r(t))|_{t=2}$. (Note: $r(t)$ is the same curve as in part (a)).

5. (13 points) Let C be the curve of intersection of the surfaces defined by the equations: $x^2 + y^2 + z^2 = 12$ and $ze^{x-y} = 2$. Find the equation of the tangent line to C at the point $(2, 2, 2)$.

6. (12 points) Let $f(x, y) = 3x^2 + 2x^3y^3 + 3y^2$.
   (a) Find and classify the critical points of f in the entire plane.
   (b) Find the minimum and maximum value of f in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$.

7. (12 points) Calculate the integrals:
   (a) $\int\int_D \frac{1}{x^2 + y^2 + 5} dxdy$ where $D = \{(x, y) : x^2 + y^2 \leq 16\}$
   (b) $\int\int_D x dxdy$ where D is the triangle whose vertices are the points $(0, 1)$, $(1, 2)$ and $(2, 0)$.

8. (12 points) Let $f(x) = \frac{e^y \cos x}{1 + x + y}$. Find the directional derivative of f at $(0, 0)$ in the direction of the vector $(4, 3)$. Find the direction of fastest ascent for f (the answer should be a unit vector).