McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 222

CALCULUS 3

Examiner: Professor W. Jonsson
Associate Examiner: Dr. S. Shahabi

Date: Friday December 11, 2009
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. This is a closed book exam. No notes, cribs sheets or books are permitted.
3. Calculators are not permitted.
4. Use of a translation dictionary is permitted.
5. Students may keep this exam paper.

This examination consists of the cover page and 1 pages of 10 questions.

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1. Test the following series for convergence (absolute or conditional) or divergence:

   (a) $\sum_{n=1}^{\infty} n^{-2}e^{n}$, (b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$.

2. (a) Find the interval of convegence of the power series $\sum_{n=2}^{\infty} \frac{(x-5)^n}{4^n(n-1)^2}$.

   (b) i. Find the Maclaurin series for

   $$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

   ii. Use this series to compute the value of $\int_0^{0.1} \frac{\sin x}{x}dx$ correct to 5 decimal places.

3. At the point with $t = 1$ on the curve $x = t, y = t^2, z = t^3$ find:

   (a) the equation of the tangent line;

   (b) the unit normal vector N and the curvature κ;

   (c) the binormal vector B.

4. Find the directional derivative of

   $$f(x, y, z) = x^3 - xy^2 - z$$

   at the point $P(1, 1, 0)$ in the direction of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

5. (a) Find the tangent plane to the surface $xyz = 27$ at the point $(1, 9, 3)$.

   (b) Find the volume of the pyramid formed by the coordinate planes and the tangent plane to the surface at $(1, 9, 3)$. (Recall that the volume of a pyramid is $\frac{1}{3}Bh$, where B is the area of its base, and h is its height.)

6. Locate the critical points of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2$ and determine whether they are local maxima or minima, or saddle points.

7. Use the Lagrange multipliers' method to find the maximum and minimum of the function $f(x, y) = x^2y^2$ subject to the constraint $9x^2 + y^2 = 9$.

8. (a) Let f be a function of two variables and suppose that all the first order partial derivatives of f exist and are continuous at all points. Show that for the composite function $w = xyf(xz, yz)$ the following equation holds:

   $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} - z\frac{\partial w}{\partial z} = 2w.$$

   (b) Suppose that the equations $F(x, y, u) = 0$ and $G(x, u, v) = 0$ define u and v as functions of the variables x and y. Show that

   $$\frac{\partial v}{\partial x} = \frac{F_xG_u - F_uG_x}{F_uG_v}, \quad \frac{\partial v}{\partial y} = \frac{F_yG_u}{F_uG_v}.$$

9. Find the volume of the solid bounded by the paraboloid $x^2 + y^2 = 4z$ and the plane $z = 4$.

10. By interchanging the order of integration, evaluate $\int_{y=0}^{\sqrt{\pi}} \int_{x=y^2}^{\pi} y \sin(x^2)\, dxdy$.