STUDENTS NAME **(underline Family name)**:
STUDENT NUMBER:
SECTION NUMBER:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 222

CALCULUS 3

Examiner: Professor M. Makkai
Associate Examiner: Professor P. Russell

Date: Thursday December 21, 2006
Time: 2:00PM - 5:00PM

INSTRUCTIONS

1. This is a closed book exam. No notes or books are permitted.
2. Calculators are not permitted.
3. All problems are of equal weight. Numerical answers may be given in the forms $\sqrt{e+\pi}$, $\cos(23°)$, $\arccos\left(\sqrt{\frac{1+\sqrt{2}}{3}}\right)$,etc.
4. Show all calculations. Give sufficient reasons for your answers.
5. Use of a regular dictionary is permitted.
6. No translation dictionaries are allowed.
7. Do not write anything on the separate pink sheet summarizing the questions.
8. This exam has eight questions. All questions carry the same weight.
9. Do all your work on the sheets provided. Do not tear or separate sheets that have been stapled together.
10. Make sure that you write your name, student number and section number at the top of this page. (If your instructor is Michael Makkai , you are in section 1; if your instructor is Peter Russell, you are in section 2.)

    This examination is comprised of a cover pages , 7 pages of questions , and 14 extra (blank) pages. There is pink sheet summarizing the problems of this final exam that you may keep.

McGILL UNIVERSITY
FINAL EXAMINATION
MATH 222 CALCULUS 3
Thursday December 21. 2006

MATH 222, Calculus 3, Fall 2006, Final Examination

1. Consider the function $f(x) = \dfrac{1}{1 - x + x^2}$.

   (a) Determine the Taylor ("Maclaurin") polynomials $T_1(x), T_2(x), T_3(x)$ of $f(x)$ centered at $a = 0$.

   (b) Determine a constant C such that $| f(x) - T_2(x) | \leq C \cdot x^3$ in the range $\frac{1}{2} \leq x \leq \frac{3}{4}$.
   (**Hints:** Show and use the fact that $1 - x + x^2 \geq \frac{3}{4}$ always. When calculating repeated derivatives, use temporary abbreviations such as the single letter s for $1 - x + x^2$.)

This page is for the continuation of problem 1; it may also be used for rough work.

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2. Consider the space curve $P(t)$ given by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 2t^2 \\ 4t \\ \ln(t^2) \end{pmatrix},$$

the parameter t varying in the range $t \geq 1$.

(a) Compute the following as functions of t:
(i) the arc length $s(t)$ of the curve from $t_0 = 1$ to t,
(ii)the unit tangent vector $\mathtt{T}(t)$,
(iii) the curvature $\kappa(t)$.

(b) Compute the principal unit normal vector $\mathbf{N}(1)$ at $t = 1$.

(**Note:** the results for (ii) and (iii) should come out as rational functions.)

This page is for the continuation of problem 2; it may also be used for rough work.

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3. Let $f(x, y) = x^3 + 2x^2y + 4y^3$

   (a) Calculate the gradient $\nabla f(2, 1)$ of the function f at the point $P = (2, 1)$.

   (b) Determine the equation of the tangent line of the level curve $f(x, y) = 20$ of f at the point $P = (2, 1)$.

   (c) Suppose that, for the unit vector $\mathbf{u} = (a, b)$, we have that the directional derivative $D_{\mathbf{u}}f(2, 1)$ equals $\frac{1}{2}|\nabla f(2, 1)|$. Determine the angle between the vectors $\nabla f(2, 1)$ and $\mathbf{u}$.

   (d) Determine all unit vectors $\mathbf{u} = (a, b)$ for which $D_{\mathbf{u}}f(2, 1) = \frac{1}{2}|\nabla f(2, 1)|$.

This page is for the continuation of problem 3; it may also be used for rough work.

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4. Let $f(x, y) = x^2y + 2xy + y^3$.

   (a) Find all the critical points of the function $f(x, y)$, and classify them. For the local maximum (maxima) and minimum (minima), determine the local maximum/minimum value of $f(x, y)$.

   (b) Consider the closed square D on the (x, y) plane defined by $-3 \leq x \leq 1, \quad -1 \leq y \leq 1$. As we know, $f(x, y)$ has an absolute maximum value M, and an absolute minimum value m on D.

   (i) Decide whether $f(x, y)$ takes the value M in the interior or on the boundary of D.
   (ii) Do the same for m.

   Give the reasons for your answers. (**Hint:** Use the values obtained in (a), but do not try to determine the values M and m; it would take too long.)

This page is for the continuation of problem 4; it may also be used for rough work.

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5. A solid in the (x, y, z)-coordinate system is bounded from below by the disk inside the circle $x^2+y^2 = 6y$ in the (x, y)-plane, and from above by a portion of the paraboloid $z = x^2+2y^2+4$. It is defined by the inequalities,

$$x^2 + y^2 \le 6y$$
$$0 \le z \le x^2 + 2y^2 + 4$$

(a) Find the inequalities that define the solid in cylindrical coordinates.

(b) The solid has a mass density whose value at (x, y, z) is $\rho(x, y, z) = y+z$. Using cylindrical coordinates, set up, ***but do not calculate***, an iterated integral equal to the total mass M of the solid.

This page is for the continuation of problem 5; it may also be used for rough work.

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6. Calculate the surface area of the part of the surface $z(x, y) = 3x^2 + \frac{y^2}{2}$ over the first-quadrant portion of the elliptical disk $6x^2 + \frac{y^2}{6} \leq 1$ that is, over the region D defined by
$6x^2 + \frac{y^2}{6} \leq 1,\ x \geq 0,\ y \geq 0.$

   Hint: to compute the double integral expressing the surface area, use the new variables r and t with the coordinate transformation

$$x = r\cos(t), \qquad y = 6r\sin(t).$$

   You will have to determine the (r, t) region P that corresponds to the given region D under the coordinate transformation.

This page is for the continuation of problem 6; it may also be used for rough work.

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7. A particle of mass 1 moves in the (x, y)-plane on the curve $x^4 + y^2 = 1$. Another particle of mass 1 is located at the origin. By Newton's law, the gravitational attraction between the particles is proportional to $\frac{1}{d^2}$, where d is the distance between the particles. Using the method of Lagrange multipliers find all the points on the curve where the attraction between the particles is largest and all points where it is the smallest.

This page is for the continuation of problem 7; it may also be used for rough work.

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SUMMARY OF FINAL EXAM FOR MATH 222

McGill University MATH 222 - Calculus 3 Final exam Fall 2006, Final Examination Questions. For your convenience on the two sides of this sheet, all problems are listed on this exam. You may keep this sheet, but do not write anything on it during the exam. All the problems are of equal weight.

Numerical answers may be given in the forms $\sqrt{e+\pi}$, $\cos(23°)$, $\arccos\left(\sqrt{\frac{1+\sqrt{2}}{3}}\right)$, etc. Show all calculations. Give sufficient reasons for your answers.

1. Consider the function $f(x) = \dfrac{1}{1-x+x^2}$.

   (a) Determine the Taylor ("Maclaurin") polynomials $T_1(x), T_2(x), T_3(x)$ of $f(x)$ centered at $a = 0$.

   (b) Determine a constant C such that $\mid f(x) - T_2(x) \mid \leq C \cdot x^3$ in the range $\dfrac{1}{2} \leq x \leq \dfrac{3}{4}$. (**Hints:** Show and use the fact that $1 - x + x^2 \geq \dfrac{3}{4}$ always. When calculating repeated derivatives, use temporary abbreviations such as the single letter s for $1 - x + x^2$.)

2. Consider the space curve given by

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   the parameter t varying in the range $t \geq 1$.

   (a) Compute the following as functions of t:
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   (c) Suppose that, for the unit vector $\mathbf{u} = (a,b)$, we have that the directional derivative $D_{\mathbf{u}}f(2,1)$ equals $\dfrac{1}{2}|\nabla f(2,1)|$. Determine the angle between the vectors $\nabla f(2,1)$ and $\mathbf{u}$.

   (d) Determine all unit vectors $\mathbf{u} = (a,b)$ for which $D_{\mathbf{u}}f(2,1) = \frac{1}{2}|\nabla f(2,1)|$.

4. Let $f(x,y) = x^2y + 2xy + y^3$.

   (a) Find all the critical points of the function $f(x,y)$, and classify them. For the local maximum (maxima) and minimum (minima), determine the local maximum/minimum value of $f(x,y)$.

   (b) Consider the closed square D on the (x,y) plane defined by $-3 \le x \le 1, \quad -1 \le y \le 1$. As we know, $f(x,y)$ has an absolute maximum value M, and an absolute minimum value m on D.

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   Hint: to compute the double integral expressing the surface area, use the new variables r and t with the coordinate transformation

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