McGill University

Faculty of Science

MATH 204

PRINCIPLES OF STATISTICS II

Final Examination

Date: 20th April 2009 Time: 9am-12pm

Examiner: Dr. David A. Stephens
Associate Examiner: Dr. Russell Steele

Please write your answers in the answer booklets provided.

This paper contains six questions. Each question carries 25 marks. Credit will be given for all questions attempted. The total mark available is 150 but rescaling of the final mark may occur.

Candidates may take one double-sided sheet of Letter-sized (216×279 mm) or A4-sized (210×297mm) paper with notes into the examination room.

Calculators may be used. Relevant statistical tables are provided.

Dictionaries and Translation dictionaries are permitted.

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1. In an experimental study of the efficacy of medication for blood pressure reduction, patients were recruited to the study and given the same amount of medication over the course of treatment. The patients were categorized by age:

- Group 1: Age 40 - 49 (group mid point 45)
- Group 2: Age 50 - 59 (group mid point 55)
- Group 3: Age 60 - 69 (group mid point 65)
- Group 4: Age 70 - 79 (group mid point 75)
- Group 5: Age 80 - 89 (group mid point 85)

SPSS output for the analysis of these data is contained on page 9: the output contains the analyses carried out by two statisticians, Analyst 1 (two pieces of output) and Analyst 2 (three pieces of output). The SPSS data set contained the following variables:

– **Age** : Age of patient in years
– **AgeGroup** : Age group to which patient belonged
– **AgeMidPoint** : Mid point of age group to which patient belonged.
– **Y** : Reduction in blood pressure over course of treatment

(a) Is this study balanced ? Comment on the relevance of balance to the analyses carried out.

2 MARKS

(b) The SPSS ANOVA table for Analyst 1 is incomplete: three key entries (marked X) are missing. Write down the values of the three missing quantities, explaining how you computed each.

6 MARKS

(c) Explain the statistical hypothesis test carried out by Analyst 1: give the null hypothesis, the test statistic and the result of the test.

5 MARKS

(d) Explain the analysis carried out by Analyst 2. What type of model is being used ? Explain the difference (in terms of the model used and the conclusions) between this analysis and the analysis carried out by Analyst 1.

8 MARKS

(e) Both Analyst 1 and Analyst 2 need to make certain assumptions in order for their statistical conclusions to be valid. List these assumptions.

4 MARKS

2. (a) An experimental study is to be carried out into the effect of two factors A (with a levels) and B (with b levels) on response Y. Explain how to carry out a complete, balanced replicated design with r replicates

   (i) if A and B are treated identically as factors,

   (ii) if B is regarded as a **blocking** factor.

   Comment in particular on how the experimental units are assigned to treatments in the two cases, and how the statistical analyses of the two designs differ.

   6 MARKS

(b) A balanced, complete experimental study with two factors A and B was carried out, and the analysis is presented on pages 10 and 11. Summarize relevant features of the SPSS output; comment in particular on

   (i) the models being fitted, and the model you would regard as the most suitable for explaining the variation in the response

   (ii) the conclusions from the ANOVA results

   (iii) the overall quality of the model fit

   (iv) whether the assumptions necessary for ANOVA results to be valid are met here.

   10 MARKS

   When describing the models, use the standard model notation such as

$$\mathsf{A} + \mathsf{B}$$

   to represent the model with main effects only, and so on.

(c) On the basis of the output given, report an estimate of the mean response for experimental units in the treatment group for

   (i) Factor A level 5, Factor B level 4.

   (ii) Factor A level 1, Factor B level 3.

   6 MARKS

(d) Explain briefly how you would proceed to carry out an analysis of a two-factor experiment that was **not** balanced

   3 MARKS

3. A data set containing information on 400 schools in California is to be used to try to understand how to predict academic performance in the school, and how to modify future education policy in the state. The data set comprises four variables:

- X_1: the academic performance in previous year (covariate)
- X_2: the average class size in current year (covariate) (two values missing)
- X_3: the total enrollment at the school in the current year (covariate)
- Y: the academic performance in current year (response)

(a) Write down the formula, involving suitable β coefficients, of a **multiple regression model** for the expected response $E[Y]$ when the model includes all three covariates as main effects.

3 MARKS

(b) Output from the SPSS analysis of the response data is given on pages 12 and 13. A summary of the model fit results are given in the following table:

Analysis	R^2	Adj. R^2	Analysis	R^2	Adj. R^2
Analysis 1	0.986	0.971	Analysis 5	0.985	0.971
Analysis 2	0.985	0.971	Analysis 6	0.171	0.029
Analysis 3	0.986	0.971	Analysis 7	0.318	0.181
Analysis 4	0.380	0.144			

Explain the results in the analyses given, first listing the models fitted for each analysis, and then summarizing the conclusions you would report. Use the standard model notation to describe the models.

10 MARKS

(c) Predict the expected academic performance for a school whose academic performance in the previous year was 700, with average class size 20, and school enrollment 650, using the results from

(i) Analysis 1,
(ii) Analysis 3,
(iii) Analysis 4.

6 MARKS

(d) Explain why you would not recommend that prediction be made using the results for Analysis 1 for a school whose academic performance in the previous year was 700, but with current average class size 28 and school enrollment 800 from these data.

2 MARKS

(e) Another variable, X_4, defined as the **change** in academic performance at the school between the previous year and the current year is also included in the data set. Explain why X_4 should **not** be used as a covariate in a model aimed at predicting Y.

4 MARKS

4. Optimal dosing of the anticoagulant drug warfarin is an important issue in the treatment of patients affected with blood clotting disorders. The optimal dose can be measured experimentally, and is thought to be dependent on the dietary level of vitamin K and genetic factors - in particular, the genotypes (*polymorphisms*) at two loci are thought to influence the level of optimal dose.

An observational study involving 107 patients was carried out to study optimal dosing. The recorded data comprise the response (the recorded optimal dose level), two factor predictors and a single continuous covariate:

- G1: a factor with three levels corresponding to locus one, corresponding to the three polymorphisms observed at that locus.
- G2: a factor with two levels corresponding to locus two, corresponding to the two polymorphisms observed at that locus.
- X: a covariate recording the dietary level of vitamin K

A series of models were fitted, and the results are summarized in a table on page 14.

(a) List the models that can be fitted using the pair of variables G1 and X only, and give the number of parameters that each of these models contains.

5 MARKS

(b) Explain, using sketches if necessary, what the model $\mathsf{G1} + \mathsf{G2} + \mathsf{X} + \mathsf{G2.X}$ represents in terms of how the mean response changes in the different subgroups. Explain the interaction term carefully.

6 MARKS

(c) Using the table of results above and a model search strategy, find the most appropriate model in ANOVA-F terms. The ANOVA-F test for comparing nested models is based on the statistic

$$F = \frac{(\mathsf{SSE_R} - \mathsf{SSE_C})/(k-g)}{\mathsf{SSE_C}/(n-k-1)}$$

- $\mathsf{SSE_R}$ is the error sum of squares for the **Reduced Model**, specified using $g+1$ parameters including the intercept.
- $\mathsf{SSE_C}$ is the error sum of squares for the **Complete Model**, specified using $k+1$ parameters including the intercept.

If the reduced model is an adequate simplification of the complete model, then

$$F \sim \text{Fisher-F}(k-g, n-k-1)$$

You do not have to perform all possible model comparisons between the eight models listed.

10 MARKS

A table of the Fisher-F distribution is provided on page 15. Entries in the table are the 0.05 tail quantile of the Fisher-F(ν_1, ν_2) distribution, for different values of ν_1 and ν_2.

(d) Report the estimate of the residual error variance σ^2 for your selected model.

2 MARKS

(e) Explain why models 3 and 4 cannot be compared using the procedure above.

2 MARKS

5. (a) A clinician has developed a new diagnostic test for early-onset Alzheimers disease, and in a pilot study was able to predict the disease correctly on 11 out of 15 occasions.

Using data obtained, the test statistic $S = 11$, and the table of the Binomial distribution on page 16, carry out a test of the hypotheses

$$H_0 \ : \ \theta = \frac{1}{2}$$

$$H_a \ : \ \theta > \frac{1}{2}$$

where θ is the probability of correctly predicting the disease.

Compute the p-value in the test, and give the critical value if the rejection region was required to contain **at most** probability 0.05.

10 MARKS

Hint: as in the sign or Binomial test, if the null hypothesis is true, $S \sim Binomial(15, 1/2)$. Also, recall that in the case of a discrete test statistic T, the one-sided rejection region is defined by considering, for critical value C_R, the probability

$$P[T \geq C_R].$$

(b) The usual test for early onset Alzheimers' is based on a genetic screening test. In a larger follow-up study of 75 patients who were ultimately determined to be sufferers from the disease, the diagnostic accuracy of the new test was compared with genetic screening, and the results are given in the following table.

Diagnosis	Test Used		
Alzheimers	New	Usual	Total
Yes	22	36	58
No	8	9	17
Total	30	45	75

Is there any evidence that diagnostic accuracy is the same for both tests ? Justify your answer by carrying out a Chi-squared test. Report also the estimate of the odds ratio for correct diagnosis derived from this table.

9 MARKS

The table on page 16 contains the 0.05 and 0.01 tail quantiles of the Chisquared(ν) distribution, for $\nu = 1$ to 20. The two forms of the Chi-squared statistic are also given.

(c) For the 75 patients in the previous study, that comprised 18 women and 57 men, the age at diagnosis was also recorded. Name two tests that can be used to assess whether there was a difference between age at diagnosis between women and men, describing briefly the assumptions that must hold for each test to be valid.

6 MARKS

6. (a) Explain the different contexts in which an ANOVA F-test, the Kruskal-Wallis test, and the Friedman test are used. Comment in particular on the relevant experimental designs, and on the assumptions needed for each test to be valid.

6 MARKS

(b) In a study of three different therapies for migraine, a small randomized study was carried out in a sample of sixteen of healthy subjects. Each subject had a migraine induced chemically, and then underwent a three hour course of therapy, at the end of which they reported their current migraine level using a score on a scale of 1 to 20, with 20 being highest pain level.

The data recorded in the study are given below.

Group	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
Score	13	5	5	10	9	6	4	12	4	12	7	15	18	12	18	15

Carry out the Kruskal-Wallis test for these data. Recall that the Kruskal-Wallis test statistic is

$$H = \frac{12}{n(n+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(n+1)$$

where R_j is the rank sum for group $j = 1, 2, 3$, and k is the number of groups being compared. If the relevant null hypothesis, H_0, is **true**, then

$$H \stackrel{\cdot}{\sim} \text{Chisquared}(k-1).$$

The table on page 16 contains the 0.05 and 0.01 tail quantiles of the Chisquared(ν) distribution, for $\nu = 1$ to 20.

10 MARKS

(c) An ANOVA test was also considered for the data in the table above. In the test for this one-way, Completely Randomized Design, the sums of squares quantities were computed to be

$$\text{SST} = 203.437 \qquad \text{SSE} = 142.000$$

Complete the Fisher-F test for the equality of the group means for these data.

5 MARKS

A table of the Fisher-F distribution is provided on page 15. Entries in the table are the 0.05 tail quantile of the Fisher-F(ν_1, ν_2) distribution, for different values of ν_1 and ν_2.

(d) For the test in (c), name a test that you would carry out in order to check one of the three assumptions underlying the ANOVA F-test ?

2 MARKS

(e) If the sample size in this study was deemed not to be large enough for the approximate null distribution of statistic H to be valid, name a type of procedure that could be used to carry out a test of the hypothesis of interest.

2 MARKS

Output for Question 1

Analyst 1

Between-Subjects Factors

		N
AgeGroup	1	13
	2	9
	3	20
	4	16
	5	17

Tests of Between-Subjects Effects

Dependent Variable:0

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	66.550[a]	4	16.637	6.089	.000
Intercept	29.820	1	29.820	10.914	.002
AgeGroup	66.550	X	16.637	X	.000
Error	191.256	70	2.732		
Total	298.640	75			
Corrected Total	X	74			

a. R Squared = .258 (Adjusted R Squared = .216)

Analyst 2

Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	.109[a]	.012	-.002	1.86804

a. Predictors: (Constant), AgeMidPoint

ANOVA[b]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	3.067	1	3.067	.879	.352[a]
	Residual	254.739	73	3.490		
	Total	257.806	74			

a. Predictors: (Constant), AgeMidPoint

b. Dependent Variable: Y

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	1.667	1.015		1.643	.105
	AgeMidPoint	-.014	.015	-.109	-.938	.352

a. Dependent Variable: Y

Output for Question 2: Part I

Model 1

Levene's Test of Equality of Error Variances[a]

Dependent Variable:Y

F	df1	df2	Sig.
1.381	19	40	.191

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.

a. Design: Intercept + A + B + A * B

Tests of Between-Subjects Effects

Dependent Variable:Y

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	210.661[a]	19	11.087	14.177	.000
Intercept	304.425	1	304.425	389.255	.000
A	170.673	4	42.668	54.558	.000
B	20.227	3	6.742	8.621	.000
A * B	19.762	12	1.647	2.106	.039
Error	31.283	40	.782		
Total	546.369	60			
Corrected Total	241.944	59			

a. R Squared = .871 (Adjusted R Squared = .809)

Parameter Estimates

Dependent Variable:Y

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	1.663	.511	3.258	.002	.631	2.695
[A=1]	.887	.722	1.228	.227	-.573	2.346
[A=2]	1.347	.722	1.865	.070	-.113	2.806
[A=3]	-2.037	.722	-2.821	.007	-3.496	-.577
[A=4]	-.313	.722	-.434	.667	-1.773	1.146
[A=5]	0[a]	.	.	.	.	.
[B=1]	.350	.722	.485	.631	-1.109	1.809
[B=2]	-.860	.722	-1.191	.241	-2.319	.599
[B=3]	-.253	.722	-.351	.728	-1.713	1.206
[B=4]	0[a]	.	.	.	.	.
[A=1] * [B=1]	3.217	1.021	3.150	.003	1.153	5.281
[A=1] * [B=2]	2.947	1.021	2.886	.006	.883	5.011
[A=1] * [B=3]	2.820	1.021	2.762	.009	.756	4.884
[A=1] * [B=4]	0[a]	.	.	.	.	.
[A=2] * [B=1]	1.843	1.021	1.805	.079	-.221	3.907
[A=2] * [B=2]	2.100	1.021	2.056	.046	.036	4.164
[A=2] * [B=3]	.297	1.021	.291	.773	-1.767	2.361
[A=2] * [B=4]	0[a]	.	.	.	.	.
[A=3] * [B=1]	1.507	1.021	1.475	.148	-.557	3.571
[A=3] * [B=2]	1.430	1.021	1.400	.169	-.634	3.494
[A=3] * [B=3]	.340	1.021	.333	.741	-1.724	2.404
[A=3] * [B=4]	0[a]	.	.	.	.	.
[A=4] * [B=1]	-.480	1.021	-.470	.641	-2.544	1.584
[A=4] * [B=2]	.223	1.021	.219	.828	-1.841	2.287
[A=4] * [B=3]	-.177	1.021	-.173	.864	-2.241	1.887
[A=4] * [B=4]	0[a]	.	.	.	.	.
[A=5] * [B=1]	0[a]	.	.	.	.	.
[A=5] * [B=2]	0[a]	.	.	.	.	.
[A=5] * [B=3]	0[a]	.	.	.	.	.
[A=5] * [B=4]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Output for Question 2: Part II

Model 2

Levene's Test of Equality of Error Variances[a]

Dependent Variable:Y

F	df1	df2	Sig.
.738	19	40	.759

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.

a. Design: Intercept + A + B

Tests of Between-Subjects Effects

Dependent Variable:Y

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	190.899[a]	7	27.271	27.782	.000
Intercept	304.425	1	304.425	310.123	.000
A	170.673	4	42.668	43.467	.000
B	20.227	3	6.742	6.868	.001
Error	51.045	52	.982		
Total	546.369	60			
Corrected Total	241.944	59			

a. R Squared = .789 (Adjusted R Squared = .761)

Model 3

Tests of Between-Subjects Effects

Dependent Variable:Y

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	170.673[a]	4	42.668	32.927	.000
Intercept	304.425	1	304.425	234.924	.000
A	170.673	4	42.668	32.927	.000
Error	71.271	55	1.296		
Total	546.369	60			
Corrected Total	241.944	59			

a. R Squared = .705 (Adjusted R Squared = .684)

Parameter Estimates

Dependent Variable:Y

Parameter	B	Std. Error	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	1.472	.329	4.481	.000	.814	2.131
[A=1]	3.133	.465	6.740	.000	2.201	4.064
[A=2]	2.407	.465	5.179	.000	1.475	3.338
[A=3]	-1.217	.465	-2.620	.011	-2.149	-.286
[A=4]	-.422	.465	-.907	.368	-1.353	.510
[A=5]	0[a]	.	.	.	.	.

a. This parameter is set to zero because it is redundant.

Output for Question 3: Part I

Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
Performance	400	369	940	647.62	142.249
Performance in Prev. Yr	400	333	917	610.21	147.136
Average class size	398	14	25	19.16	1.369
Enrollment	400	130	1570	483.47	226.448
Valid N (listwise)	398				

Analysis 1

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	75.968	17.173		4.424	.000
	Performance in Prev. Yr	.945	.009	.978	106.815	.000
	Average class size	.137	.912	.001	.151	.880
	Enrollment	-.015	.006	-.024	-2.621	.009

a. Dependent Variable: Performance

Analysis 2

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	72.043	17.234		4.180	.000
	Performance in Prev. Yr	.952	.008	.986	113.116	.000
	Average class size	-.275	.905	-.003	-.304	.761

a. Dependent Variable: Performance

Analysis 3

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	77.599	6.700		11.581	.000
	Performance in Prev. Yr	.946	.009	.978	109.902	.000
	Enrollment	-.015	.006	-.023	-2.630	.009

a. Dependent Variable: Performance

Analysis 4

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	337.602	92.915		3.633	.000
	Average class size	21.604	4.861	.208	4.444	.000
	Enrollment	-.213	.029	-.341	-7.278	.000

a. Dependent Variable: Performance

Output for Question 3: Part II

Analysis 5

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	66.326	5.189		12.783	.000
	Performance in Prev. Yr	.953	.008	.985	115.236	.000

a. Dependent Variable: Performance

Analysis 6

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	308.337	98.731		3.123	.002
	Average class size	17.751	5.140	.171	3.454	.001

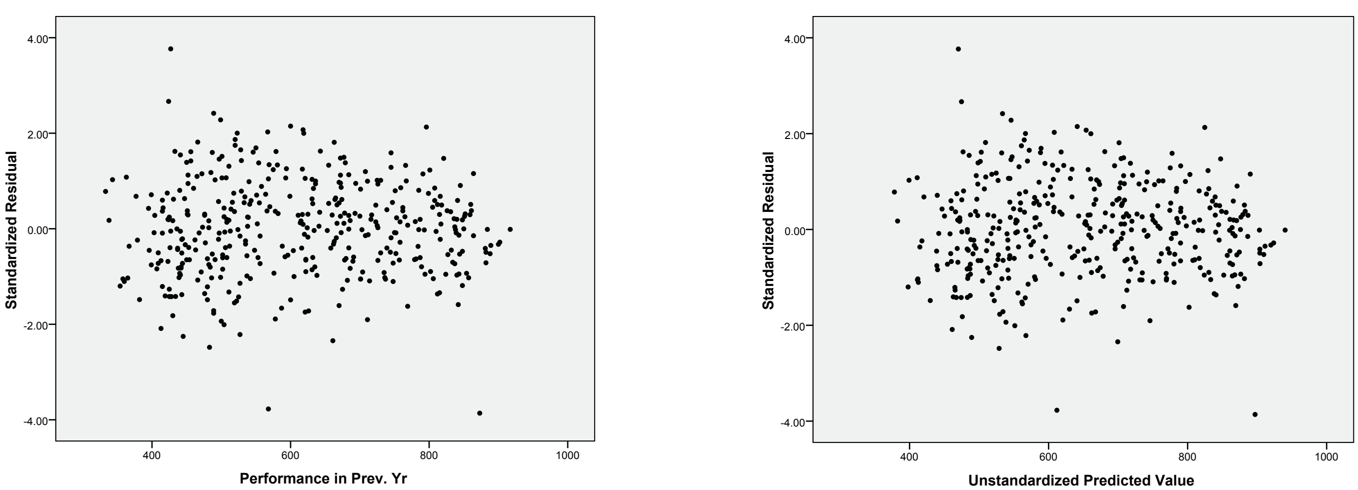
a. Dependent Variable: Performance

Analysis 7

Coefficients[a]

Model		Unstandardized Coefficients B	Unstandardized Coefficients Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	744.251	15.933		46.711	.000
	Enrollment	-.200	.030	-.318	-6.695	.000

a. Dependent Variable: Performance



Residual plots from fit in Analysis 1

Table for Question 4

Note: p is the total number of parameters including the intercept.

Model	Formula	SSE	p
1	G1 $*$ G2 $*$ X	16.787	12
2	G1 + G2 + X + G1.G2 + G1.X + G2.X	16.813	10
3	G1 + G2 + X + G1.G2 + G1.X	16.936	9
4	G1 + G2 + X + G1.G2 + G2.X	17.516	8
5	G1 + G2 + X + G1.X + G2.X	30.992	8
6	G1 + G2 + X + G1.G2	17.593	7
7	G1 + G2 + X	33.121	5
8	G1 + G2 + G1.G2	253.246	6

Table of the Fisher-F distribution

Entries in table are the $\alpha = 0.05$ tail quantile of Fisher-F(ν_1, ν_2) distribution
ν_1 given in columns, ν_2 given in rows.

$\nu_2 \backslash \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	242.98	243.91	244.69	245.36	245.95	246.46	246.92	247.32	247.69	248.01	248.31	248.58	248.83	249.05	249.26
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40	19.41	19.42	19.42	19.43	19.43	19.44	19.44	19.44	19.45	19.45	19.45	19.45	19.45	19.46
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68	8.67	8.67	8.66	8.65	8.65	8.64	8.64	8.63
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81	5.80	5.79	5.79	5.78	5.77	5.77
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57	4.56	4.55	4.54	4.53	4.53	4.52
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88	3.87	3.86	3.86	3.85	3.84	3.83
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46	3.44	3.43	3.43	3.42	3.41	3.40
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16	3.15	3.14	3.13	3.12	3.12	3.11
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95	2.94	2.93	2.92	2.91	2.90	2.89
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79	2.77	2.76	2.75	2.75	2.74	2.73
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66	2.65	2.64	2.63	2.62	2.61	2.60
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56	2.54	2.53	2.52	2.51	2.51	2.50
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47	2.46	2.45	2.44	2.43	2.42	2.41
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40	2.39	2.38	2.37	2.36	2.35	2.34
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34	2.33	2.32	2.31	2.30	2.29	2.28
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29	2.28	2.26	2.25	2.24	2.24	2.23
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24	2.23	2.22	2.21	2.20	2.19	2.18
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20	2.19	2.18	2.17	2.16	2.15	2.14
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17	2.16	2.14	2.13	2.12	2.11	2.11
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14	2.12	2.11	2.10	2.09	2.08	2.07
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11	2.10	2.08	2.07	2.06	2.05	2.05
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08	2.07	2.06	2.05	2.04	2.03	2.02
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06	2.05	2.04	2.02	2.01	2.01	2.00
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04	2.03	2.01	2.00	1.99	1.98	1.97
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02	2.01	2.00	1.98	1.97	1.96	1.96
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00	1.99	1.98	1.97	1.96	1.95	1.94
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93	1.92
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.93	1.92	1.91	1.91
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96	1.94	1.93	1.92	1.91	1.90	1.89
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95	1.93	1.92	1.91	1.90	1.89	1.88
31	4.16	3.30	2.91	2.68	2.52	2.41	2.32	2.25	2.20	2.15	2.11	2.08	2.05	2.03	2.00	1.98	1.96	1.95	1.93	1.92	1.91	1.90	1.88	1.88	1.87
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19	2.14	2.10	2.07	2.04	2.01	1.99	1.97	1.95	1.94	1.92	1.91	1.90	1.88	1.87	1.86	1.85

Table of the Binomial$(15, 1/2)$ distribution

s	0	1	2	3	4	5	6	7	8	9	10
$P[S = s]$	0.0000	0.0005	0.0032	0.0139	0.0417	0.0916	0.1527	0.1964	0.1964	0.1527	0.0916
$P[S \leq s]$	0.0000	0.0005	0.0037	0.0176	0.0592	0.1509	0.3036	0.5000	0.6964	0.8491	0.9408

s	11	12	13	14	15
$P[S = s]$	0.0417	0.0139	0.0032	0.0005	0.0000
$P[S \leq s]$	0.9824	0.9963	0.9995	1.0000	1.0000

Table of the Chisquared(ν) distribution

Entries in table are the $\alpha = 0.05$ and $\alpha = 0.01$ tail quantiles

ν	1	2	3	4	5	6	7	8	9	10
$\alpha = 0.05$	3.841	5.991	7.815	9.488	11.070	12.592	14.067	15.507	16.919	18.307
$\alpha = 0.01$	6.635	9.210	11.345	13.277	15.086	16.812	18.475	20.090	21.666	23.209

ν	11	12	13	14	15	16	17	18	19	20
$\alpha = 0.05$	19.675	21.026	22.362	23.685	24.996	26.296	27.587	28.869	30.144	31.410
$\alpha = 0.01$	24.725	26.217	27.688	29.141	30.578	32.000	33.409	34.805	36.191	37.566

The Chi-squared statistic takes one of the two forms depending on the test being carried out.

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \qquad\qquad X^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(n_{ij} - \widehat{n}_{ij})^2}{\widehat{n}_{ij}}$$