McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 150

CALCULUS A

Examiner : Dr. Junfang Li
Associate Examiner : Professor Axel Hundemer

Date : December 7, 2007
Time : 2 : 00pm – 5 : 00pm

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.

2. Give detailed and complete solutions and fully simplify your answers.

3. Calculators are not permitted.

4. This is a closed book exam.

5. Use of a regular dictionary is not permitted.

This exam comprises the cover page, and 2 pages of 6 questions.

1. (18 marks) Note : there is no relation between (a), (b), (c) and (d). Fully justify EACH STEP!

   (a) Evaluate the following limit without using L'Hôpital's rule.

   $$\lim_{x\to 3} \frac{1-\cos(x-3)}{x-3}$$

   (b) Evaluate the following limit or show that it doesn't exist.

   $$\lim_{x\to 3} \frac{\sqrt{x+3}-\sqrt{2x}}{x^2-3x}$$

   (c) Suppose that $g(x,y) = \frac{x^4-y^4}{x^4+x^2y^2+y^4}$, if $(x,y) \neq (0,0)$; we define $g(0,0)$ to be zero. Show that g is not continuous at $(0,0)$.

   (d) Suppose that $f(x,y) = (\sin x + y)^{y \sin x}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

2. (12 marks) Consider the curve C defined by the equation $\sin^2 x + \cos^2 y = 1$.

   (a) Find the equation of the tangent line to C at the point $P(\frac{\pi}{4}, \frac{\pi}{4})$.

   (b) Compute $\frac{d^2y}{dx^2}$ at P.

3. (16 marks) Let $f(x) = \frac{x^2-2}{x^3}$.

   (a) Find the domain of f. Determine x-intercepts and y-intercepts (if any). Determine all horizontal and vertical asymptotes of f (if any).

   (b) Find and classify the critical and singular points (if any) and find the intervals of increase and decrease of f.

   (c) Find all inflection points (if any) and the intervals of concavity of f.

   (d) Using all the information obtained above, carefully sketch the graph of f.

4. (16 marks) For each of the following series determine whether it is absolutely convergent, conditionally convergent or divergent. Fully justify your answers!

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\arctan n)^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{e^{n^2}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(e^n)}{n\sqrt{n^2+1}}$

5. (22 marks) Let $f(x,y) = (x^2+y^2)e^{x^2-y^2}$.

   (a) Find the equation for the tangent plane to the surface given by $f(x,y)$ at the point $P(1,1,2)$.

   (b) Find the directional derivative $D_{\mathbf{u}}f$ at P, where $\mathbf{u}$ is the unit vector in the direction of $\langle 1,1\rangle$.

   (c) Find and classify all critical points of f.

   (d) Find the maximum and minimum values of f on the disk $x^2+y^2 \leq 4$.

6. (21 marks) Suppose that $\omega = \omega\big(r(x,y),t\big)$ is defined by $\omega = \frac{1}{r}f(t-r)$ and that $r(x,y) = \sqrt{x^2+y^2}$. x, y, and t are independent variables, and f is differentiable.

   (a) Show that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Find a similar equation for $\dfrac{\partial r}{\partial y}$.

   (b) Show that $\dfrac{\partial^2 r}{\partial x^2} = \dfrac{1}{r} - \dfrac{x^2}{r^3}$. Find a similar equation for $\dfrac{\partial^2 r}{\partial y^2}$.

   (c) Show that $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}$.

   (d) Show that $\dfrac{\partial \omega}{\partial x} = -\dfrac{x}{r^3}f(t-r) - \dfrac{x}{r^2}f'(t-r)$.

   (e) Find $\dfrac{\partial^2 \omega}{\partial x^2}$.

   [Fully justify each step.]