Final Examination December 14, 1998 189-133A

Group 3

PART I: Multiple Choice Questions

1. For the linear system $\begin{pmatrix} 2+a & 1-a \\ 4+3a & 3+a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

   (a) When $a \neq \frac{-1}{2}$ and $a \neq -1$, there are infinitely many solutions.

   (b) When $a \neq \frac{-1}{2}$, there is a unique solution.

   (c) In both the cases $a = \frac{-1}{2}$ and $a = -1$, there are infinitely many solutions.

   (d) When $a = \frac{-1}{2}$, there are infinitely many solutions.

   (e) When $a = -1$, there are infinitely many solutions.

2. For a 3×3 matrix A, we have $A^3 = -\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. What is $(A - \mathbf{I})^{-1}$?

   (a) $A^2 - A + \mathbf{I}$

   (b) $A - \mathbf{I}$ might not be invertible

   (c) $\frac{-1}{2}(A^2 + A + \mathbf{I})$

   (d) $\frac{1}{2}(A^2 - A + \mathbf{I})$

   (e) $\frac{-1}{2}\mathbf{I}$

3. The point $(-3, 2, 1)^t$ is on both of the planes

$$x + y + 3z = 2$$

and

$$x - y + z = -4.$$

Which of the following gives all the points of intersection?

(a) $\frac{x+3}{2} = \frac{y-2}{1} = \frac{z-1}{-1}$,

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$,

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$,

(d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$,

(e) Only the point $(-3, 2, 1)^t$.

5. Which one is a subspace of $\mathbf{R}^4$?

(a) $\{(a,b,c,d)^t \in \mathbf{R}^4 \mid a^2 = b^2 \ , \ c+d=0\}$

(b) $\{(a,b,c,d)^t \in \mathbf{R}^4 \mid a+b=0 \ , \ c+d=0\}$

(c) $\{\mathbf{v} \in \mathbf{R}^4 \mid \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 0 & 3 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$

(d) $\{(a,b,c,d)^t \in \mathbf{R}^4 \mid a^2+b^2 > 0\}$

(e) all vectors that are *not* orthogonal to $(1,0,0)^t$

6. What are the coordinates of the vector $(1,1,1)^t$ with respect to the ordered basis $\{(1,0,0)^t, (1,1,0)^t, (1,1,1)^t\}$?

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

7. Suppose $\mathbf{v_1}, \ldots, \mathbf{v_6}$ are vectors in $\mathbf{R}^4$ and we write them as the columns of a 4×6 matrix A. Upon doing some elementary row operations, we find a row echelon form of A is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Which of the following gives a basis for span$\{\mathbf{v_1}, \ldots, \mathbf{v_6}\}$?

(a) $\{\mathbf{v_1}+\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$

(b) $\{\mathbf{v_1}, \mathbf{v_4}-\mathbf{v_3}, \mathbf{v_6}\}$

(c) $\{(1,0,0,0)^t, (4,1,0,0)^t, (6,1,1,0)^t\}$

(d) $\{\mathbf{v_1}, \mathbf{v_3}, \mathbf{v_6}\}$

(e) $\{\mathbf{v_1}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_6}\}$

8. Suppose a_{ij} are chosen so that $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 2$.

What is the determinant of $\begin{pmatrix} a_{11}+3a_{21} & a_{12}+3a_{22} & a_{13}+3a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1}$?

9. The vectors $\mathbf{u}_1 = (1,0,1)^t$, $\mathbf{u}_2 = (1,1,1)^t$, $\mathbf{u}_3 = (0,1,1)^t$ form a basis for $\mathbf{R}^3$. Applying the Gram-Schmidt orthonormalization process to this ordered basis yields the following orthonormal basis of $\mathbf{R}^3$.

(a) $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}. \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$, (b) $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$,

(c) $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, (d) $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

(e) $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

10. A conic in the xy plane is given by the equation $4x^2 - 2xy + 4y^2 = 3$. After making the change of variables $\begin{pmatrix}x\\y\end{pmatrix} = P\begin{pmatrix}X\\Y\end{pmatrix}$, where $P = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$, the conic has the equation

(a) $4X^2 - 2XY + 4Y^2 = 3$,
(b) $3X^2 + 5Y^2 = 3$,
(c) $6X^2 + 10Y^2 = 3$,
(d) $5X^2 + 3Y^2 = 3$,
(e) $10X^2 + 6Y^2 = 3$.

11. Let

$$A = \begin{pmatrix}1 & -3 & 3\\-3 & 1 & -3\\3 & -3 & 1\end{pmatrix}.$$

Which of the following is true?

(a) $(0,0,0)^t$ is an eigenvector for the eigenvalue 0 of A
(b) $(1,1,1)^t$ is an eigenvector for eigenvalue 7 of A
(c) A has no eigenvectors
(d) $(1,1,0)^t$ and $(-1,0,1)^t$ are both eigenvectors for the eigenvalue -2 of A
(e) $(1,-1,1)^t$ is an eigenvector for eigenvalue 1 of A

PART II:

1. (16 marks) The solution set of the equation $8x^2 + 4xy + 5y^2 = 36$ gives a conic C in the xy coordinate system.

   (a) Find an orthogonal matrix P which brings the conic C to standard form in the XY coordinate system with the change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{bmatrix} X \\ Y \end{bmatrix}$.

   (b) What is the equation of conic C in the new XY coordinate system? What kind of conic is C?

   (c) Make a sketch showing the relation between the XY and xy coordinate systems *and* draw the conic C in the original xy coordinate system.

Continuation page for question 1.

2. (18 marks) Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 3 & -2 & 5 & 1 & -6 \\ 2 & -4 & 6 & 6 & 4 \\ 2 & -5 & 7 & 8 & 7 \end{pmatrix}.$$

Performing Gaussian elimination on A for a few steps yields the matrix

$$A' = \begin{pmatrix} 1 & -1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & -2 & 2 & 4 & 6 \\ 0 & -3 & 3 & 6 & 9 \end{pmatrix}.$$

The book-keeping matrix at this point is

$$B' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Continuing from this point, use the Gaussian elimination algorithm while continuing to keep track of the book-keeping matrix to obtain the reduced row echelon form $\widehat{A}$ of A, and the final book-keeping matrix B (i.e. start your row reduction on A' with current book-keeping matrix B').

(b) Find a basis for the row space of A.

(c) Find all dependence relations on the columns of A.

(d) Find all dependence relations on the rows of A.

(e) Find a basis for the column space of A consisting of columns of A.

(f) Determine if the vector $(1, -1, 1, -1, 1)$ is in the row space of A.

Continuation page for question 2.

Continuation page for question 2.

3. (12 marks) Consider the two lines

$$L_1: \quad \frac{x-2}{1} = \frac{y-2}{2} = \frac{z-2}{1}$$

and

$$L_2: \quad \frac{x-5}{3} = \frac{y-6}{2} = \frac{z}{-1}.$$

(a) Find the equation of the plane which contains L_1 and is parallel to L_2 in two different forms:

i. cartesian form (i.e. $ax + by + cz + d = 0$)

ii. vector parametric form (i.e. $\mathbf{v} = \overrightarrow{OP} + s\mathbf{d_1} + t\mathbf{d_2}$)

(b) Find the shortest distance from the point $(3,3,2)^t$ to the line L_2.

Continuation page for question 3.