Group 1

PART I: Multiple Choice Questions

1. The vertices of a parallelogram are $A(1,2,0)$, $B(5,4,0)$, $C(2,1,0)$ and $D(6,3,0)$. Using the vectors $\vec{AB}$ and $\vec{AC}$ the area is computed to be

   (a) -6, (b) 22, (c) 6, (d) -22, (e) 46.

2. Consider the plane containing the points $P(1,2,0)$, $Q(1,1,1)$ and $R(3,-1,1)$. The equation of a line orthogonal to this plane is

   (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, (d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

   (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,

   (c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

3. Which of the following four matrices has a unique solution for its associated homogeneous system of equations?

   (i) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, (iii) $\begin{pmatrix} 5 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 4 & 2 \end{pmatrix}$, (iv) $\begin{pmatrix} 1 & 1 & 8 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix}$

   (a) (i) and (iv), (b) (i),(ii) and (iv), (c) (ii) and (iii), (d) all of them, (e) none of them.

4. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Then the vector $\mathbf{u} - \text{Proj}_{\mathbf{d}}(\mathbf{u})$ is orthogonal to $\mathbf{d}$ and is equal to

   (a) $1/14 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, (b) $1/14 \begin{bmatrix} 10 \\ 15 \\ -5 \end{bmatrix}$, (c) $1/14 \begin{bmatrix} 56 \\ 14 \\ 0 \end{bmatrix}$,

   (d) $1/14 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, (e) $1/14 \begin{bmatrix} 46 \\ -29 \\ 5 \end{bmatrix}$.

5. Which of the following subsets of $\mathbf{R}^2$ are subspaces?

6. Consider the following bases for $\mathbf{R}^2$

$$T = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad T' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

The transition matrix from T' to T is

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

7. Which of the following equations has as its roots the eigenvalues of the matrix (i.e. the characteristic equation or its negative).

$$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $\lambda^3 - 4\lambda^2 - 1\lambda - 1 = 0$

(b) $\lambda^3 - 4\lambda^2 - 2\lambda - 2 = 0$

(c) $-6\lambda^3 - 1\lambda^2 - 1\lambda - 4 = 0$

(d) $\lambda^3 - 6\lambda^2 - 2\lambda - 2 = 0$

(e) $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

8. Let $\mathbf{b_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b_3} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. The coordinate vector of $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ with respect to the basis $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ is

(a) $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, (b) $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, (c) $\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}$, (d) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, (e) $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$.

9. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$. A basis for the nullspace (or kernel) of A is

(a) $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, (b) $\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, (c) $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, (d) $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, (e) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

10. Let $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{3}$. The determinant of C^{-1} is

PART II: Long Answer Questions

1. Let A be the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ -1 & 7 & -1 & 3 \\ -2 & 4 & 3 & 6 \\ 3 & 4 & -2 & 6 \end{bmatrix}.$$

   (a) Reduce to reduced row echelon form, keeping track of the bookkeeping matrix.

(b) Find the dependence relations on the rows of A.

(c) Find a basis for the column space of A from among the columns of A.

(d) Determine whether $\mathbf{b} = \begin{bmatrix} 7 \\ -5 \\ 3 \\ 8 \end{bmatrix}$ belongs to the column space of A.

2. Consider the four points $A(2, 0, -7), B(1, -1, -8), C(5, 1, -5), D(3, 1, -4)$ in R^3.

   (a) Find an equation of the plane containing the points A, B and C;

   (b) Find the distance from point D to the plane;

   (c) Evaluate the area of triangle ABC.

Question 2 continued:

3. The equation $5x^2 - 2\sqrt{3}xy + 7y^2 = 16$ represents a conic in the xy coordinate system.

(a) Find a rotation of co-ordinates that bring the conic into standard form.
What is the matrix of this rotation?
What is the cosine of the angle of rotation?

(b) Make a sketch showing the relation between the two co-ordinate systems. Identify the type of conic and make a sketch of the conic in the rotated coordinate system.