MATH 325 – ORDINARY DIFFERENTIAL EQUATIONS

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Based on lectures by Tony Humphries

CONTENTS

Some things of note:

All general ODE forms will be framed,

$$\boxed{L[y] = g(x)}$$

and all important theorems will be bubbled.

$$a^2 + b^2 = c^2$$

We will be using linear differential operators throughout, denoted by $L[y]$, which "act" on equations resembling

$$y = a(x)D^n + b(x)D^{n-1} + ... + z(x)$$

where D is the derivative.

For instance, homogeneous equations appear as $L[y] = 0$ and non-homogeneous equations as $L[y] = g(x)$.

"The proof for this will, of course, be left as an exercise"

- Professor Hundemer

I First Order Equations

Here we go! First order differential equations are any equation which contains first order derivatives. Just the same, if an equation contains *any* derivatives, it is an ODE.

LINEAR

Homogeneous

To be "homogeneous" is to have all terms containing y or its derivatives sum to 0. Thus, consider the homogeneous ODE of the form

$$\boxed{y' + p(x)y = 0}$$

Here, the general solution for $y(x)$ will be

$$y(x) = Ce^{-\int p(x)dx}$$

where C may be determined by plugging in initial values, i.e. $y(x_0) = y_0$. The proof for this, like that of many formulas to come, can be found elsewhere.

Non-Homogeneous

A non-homogeneous ODE is of the similar form

$$\boxed{y' + p(x)y = g(x)}$$

Here is a useful algorithm for finding $y(x)$:

1. Let $u(x) = e^{\int p(x)dx}$
2. Multiply the given ODE by $u(x)$, yielding
$$u(x)y' + u(x)p(x)y = g(x)u(x)$$
3. By math magic, this will always simplify to
$$\frac{d}{dx}[u(x)y] = g(x)u(x)$$
4. You can thus integrate to get
$$u(x)y = \int g(x)u(x)dx$$
5. And finally
$$y = \frac{\int g(x)u(x)dx}{u(x)}$$

...making sure to mind the C, when it comes up. This general form is tedious to remember and employ, so using $u(x)$, or an *integrating factor*, is best practice.

NONLINEAR

The general form of a first order, nonlinear ODE is $f(x,y) = y'$ (but this does not imply that the equation will be written explicitly).

Separable Equations

Our first nonlinear ODE may be familiar from your last calculus class:

$$\boxed{\frac{dy}{dx} = P(x)Q(y)}$$

By rearranging each term such that dx and $P(x)$ remain on one side of the equation, while dy and $Q(y)$ remain on the other, you can solve by integrating as follows:

$$\left[\frac{1}{Q(y)}\right]dy = P(x)dx \rightarrow \int \frac{1}{Q(y)}dy = \int P(x)dx$$

Two constants will come out of each side—let's call them k_1 and k_2—and these can hence be combined to form one constant.

Exact Equations

Exact ODEs look like

- $Mdx + Ndy = 0$

  or

- $M + N\left[\frac{dx}{dy}\right] = 0$

  or

- $M\left[\frac{dy}{dx}\right] + N = 0$

These are functionally identical forms, with the first equation being found often in the wild. It may be tricky to spot one of the last two if you're not expecting it, though, as M and N may be functions of x, y, or both.

Very Exact

When considering an equation of one of the above forms, the following condition will help us immensely:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If this does not turn out to be true, we'll need to do a little more work. For now, though, let's assume it is. Then, we have the following relation by math magic:

$$M = \frac{\partial f}{\partial x} \text{ and } N = \frac{\partial f}{\partial y}$$

where $f(x, y) = C$, which can be found by integrating, satisfies the ODE.

NOTE: When you integrate M and N, you will end up with two constants, $k_1(y)$ for M and $k_2(x)$ for N, and these must be chosen such that $f_1(x, y)$, derived from M, and $f_2(x, y)$, derived from N, are equivalent.

Not Quite Exact

If, when you find M_y and N_x, they are not equivalent, it is possible to *make* them equivalent using integrating factors. First, though, one the following conditions must hold:

1. $\frac{1}{N}(M_y - N_x)$ is a function of x only

or

2. $\frac{1}{M}(M_y - N_x)$ is a function of y only

In the first case, let $u(x) = e^{\int \frac{M_y - N_x}{N} dx}$. Otherwise, let $u(y) = e^{-\int \frac{M_y - N_x}{M} dy}$. If neither are true, then we give up.

Like our method for finding the solution to non-homogeneous linear ODEs, multiplying our original equation by $u(x)$ or $u(y)$ will yield a "nicer" problem. In this context, "nice" means that the multiplied equation is now exact!

Thus, let $\tilde{M} = u \cdot M$ and $\tilde{N} = u \cdot N$. Our new, exact equation is now

$$\tilde{M} dx + \tilde{N} dy = 0$$

which we can solve.

Pseudo-Homogeneous Exact

ODEs as a subject has a major oversight, being that it works with two entirely unrelated meanings of "homogeneous." For our purposes, "homogeneous ODE" will retain its base definition, that, in terms of differential operators, $L[y] = 0$. Pseudo-homogeneity will then be defined as

$$\boxed{N(tx, ty) = t^d N(x, y) \text{ and } M(tx, ty) = t^d M(x, y)}$$

where N and M are defined exactly as above.

If we have an ODE of the form $M dx + N dy = 0$ that is neither exact nor capable of being made exact through integrating factors, it may be solved if M and N are pseudo-homogeneous. In that event, let $y = u(x) \cdot x$ and make any appropriate substitutions. The resulting ODE, $u' = f(x, u)$, will be separable.

ABC Substitution

Consider a nonlinear ODE of the form

$$\boxed{y' = f(Ax + By + C) + k}$$

where f may be any function (ex. $e^{Ax+By+C}$ or $[Ax + By + C]^3$) and $B \neq 0$.

A useful substitution in this case is $u = Ax + By + C$, which, when simplified, will yield a separable equation.

Bernoulli Equations

Consider an ODE of the form

$$\boxed{y' + p(x)y = g(x)y^n}$$

This is a Bernoulli equation, and can be made into a much easier-to-solve linear equation by using the following substitution:

$$v(x) = y^{1-n}$$

After deriving a solution to the linear ODE, $v(x)$, reversing the substitution will give you a final expression for $y(x)$, though some rearranging may be necessary. Note that, for $n = 0$ and $n = 1$, the corresponding Bernoulli equation is automatically linear.

When considering a Bernoulli IVP, it's crucial to evaluate your constant *after* undoing the substitution.

Picard Iteration

No matter the form of our nonlinear ODE, we may try solving it via Picard Iteration, which, analogous to Taylor Series for functions, will provide an increasingly accurate estimation for $y(x)$. Consider the following relation, where $f(x, y) = y'$ and $y(x_0) = y_0$:

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

The expression $f(t, y_0(t))$ may seem a little confusing at first. To clarify, here is a brief algorithm for finding it:

1. For your first iteration, let $y_n(t)$ be defined as y_0, the initial condition.

2. In the expression $f(x, y)$, replace all dependent variables y with the initial condition, and swap all x's with t's.

3. Evaluate y_{n+1} to find your first iteration, and call it y_1.

4. For your next iteration, follow steps 1-3, with any dependent variables in $f(x,y)$ being replaced by y_1 instead of $y(x_0)$.

Occasionally, a pattern will begin to emerge after repeated iterations, which, though not rigorously, may suggest a general, summation form for $y(x)$.

EXISTENCE AND UNIQUENESS FOR FIRST ORDER ODES

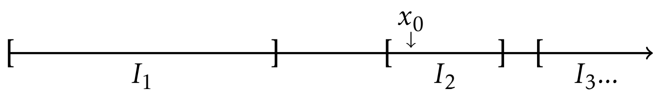
Intervals of Validity

Often, there are discontinuities in an ODE's solution and its derivative. The IOV for a particular solution depicts the "space" around an initial condition such that these discontinuities are excluded, i.e. the solution is continuously differentiable and the initial condition $y(x_0) = y_0$ is satisfied.

Before finding the appropriate IOV, it is useful to make a list of *possible* intervals by eliminating certain values for x:

- If $L[y]$ is linear, i.e. of the form $y' + p(x)y = g(x)$, the solution is continuously differentiable in x so long as $p(x)$ and $g(x)$ are continuous in x. Find these discontinuities, and remove them from the real line to yield a collection of valid intervals. This means that a solution to the ODE is not required to find its interval of validity.
- If $L[y]$ is nonlinear, with $y' = f(x,y)$, we first have to show that there exists a unique solution for a given initial condition[1], the process for which is shown in the next section. Once this is established, discontinuities in $y(x)$ may simply be removed from the real line to yield possible intervals.

Once a set of intervals is found, the one which contains x_0 is the appropriate interval of validity, as shown below.



I_2 = the interval of validity for x_0

Picard–Lindelöf Theorem

For nonlinear ODEs, finding $y(x)$ does not necessarily imply that it is the *only* solution. To prove the existence of a unique solution, we first must consider $f(x,y)$ and $\frac{\partial f}{\partial y}$ at $y(x_0) = y_0$.

> **If** $f(x,y)$ is continuous at x_0 and $\frac{\partial f}{\partial y}$ is continuous at y_0, **then** there exists a unique solution to the ODE at $y(x_0) = y_0$.

This is called the Picard–Lindelöf theorem, the proof for which brought about Picard iteration.

[1]In the case of linear ODEs, it is always true that a unique solution exists.

Lipschitz Continuity

While the above theorem proves the existence and uniqueness of a solution just fine, it is slightly overpowered. A special case of the "weaker" condition, Lipschitz continuity, is sufficient. In ODEs, a function is Lipschitz continuous on y if

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$$

for any $x, y_1, y_2 \in \mathbb{R}$ and positive constant L.

Only certain Lipschitz constants, however, will truly satisfy the theorem. Let's define the following:

$$R = x \in [x_0 - h, x_0 + h] \text{ and } y \in [y_0 - b, y_0 + b]$$

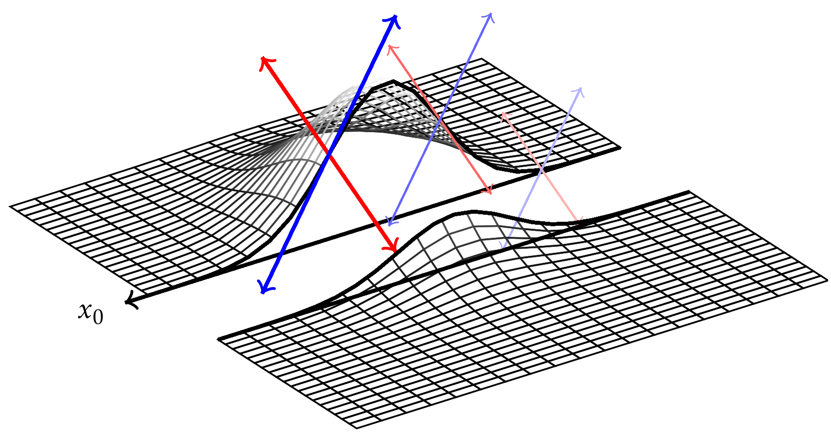
$$L = \max_{(x,y)\in R} \left| \frac{\partial f}{\partial y} \right|$$

$$M = \max_{(x,y)\in R} |f(x, y)|$$

If the inequalities below are satisfied, then so is Picard–Lindelöf:

$$Lh \le 1 \text{ and } Mh \le b$$

In general, continuity of $\frac{\partial f}{\partial y}$ on y implies Lipschitz continuity of $f(x, y)$ on y, which, as a stronger form of continuity, implies ordinary continuity of $f(x, y)$ on y. These relations are strictly one-way. Below is a visualization of Lipschitz continuity at a fixed "slice" of a function:



II Second Order Equations

CONSTANT COEFFICIENT

As it turns out, it is quite difficult to find solutions to second order ODEs that diverge from a select few general forms, so for the next few pages we'll only consider equations whose derivatives are multiplied by real-valued constants.

Homogeneous

Second order, linear, constant coefficient, homogeneous ODE. That's a lot of words to describe a class of equations. Thankfully, though, they look pretty simple. Consider the following equation:

$$\boxed{ay'' + by' + cy = 0}$$

where a, b, and c are all real-valued constants. To find an expressions for $y(x)$, we first have to derive the "characteristic equation":

$$ar^2 + br + c = 0$$

The roots that come out of this equation may be real, complex, or repeated. Our "guess" for $y(x)$ will depend on the type of roots of the characteristic equation.

Real Roots

Let r_1 and r_2 be two real-valued roots of a given characteristic equation. Then, the general form for $y(x)$ is as follows:

$$y_{\mathfrak{B}}(x) = c_1 e^{r_1 x} + c_2 e^{r_1 x}$$

Note that both $c_1 e^{r_1 x}$ and $c_2 e^{r_1 x}$ themselves are solutions to the ODE. In general, second order ODEs will often have 2 or more solution that, together, form a "basis of the solution space." Now, and in future sections, we'll refer to these solutions as "complementary" solutions.

Complex Roots

In the event that your characteristic equation has complex-valued roots[2], the following will describe your basis of the solution space:

$$y_{\mathfrak{B}}(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

where α describes the Re part of your root and β describes the Im part.

Note that your set of roots, if complex, will follow the general form $\alpha \pm \beta i$

[2] If one root is complex, it is especially true that both roots are complex. You can convince yourself of this by giving the quadratic equation a good stare.

Repeated Roots

Finally, consider the case where your characteristic equation has a pair of repeated roots, i.e. $(r - r_r)^2 = 0$. Then, the solution space is as follows:

$$y_{\mathfrak{B}}(x) = c_1 e^{r_r x} + c_2 x e^{r_r x}$$

where r_r is your repeated root.

Non-Homogeneous

Undetermined Coefficients

The only difference between the class of equations described above and these are, notably, the $g(x)$ that appears on the right-hand side. For the sake of formality, here is what that looks like:

$$\boxed{ay'' + by' + cy = g(x)}$$

The general solution (i.e. basis of the solution space) for this class of equations is

$$y_{\mathfrak{B}}(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

where the first two terms are complimentary solutions that can easily be derived from the characteristic polynomial. The trouble comes, then, in the $y_p(x)$, or the "particular solution," which is solely dependent on the $g(x)$ term. There are limitations in finding explicit particular solutions, notably that the $g(x)$ from which it arises must be of a specific form. These forms are listed below, along with the appropriate "guess" for what might be our particular solution.

$$g(x) = \begin{cases} \alpha \cdot e^{\beta x} \\ \alpha \cdot \cos(\beta x) \\ \alpha \cdot \sin(\beta x) \\ \text{Any } n^{th} \text{ degree polynomial} \end{cases} \quad \rightarrow \quad y_p(x) = \begin{cases} A \cdot e^{\beta x} \\ A \cdot \cos(\beta x) + B\sin(\beta x) \\ A \cdot \cos(\beta x) + B\sin(\beta x) \\ Ax^n + Bx^{n-1} + \ldots + Yx + Z \end{cases}$$

From here, I should note a couple tricks:

For example: $g(x) = 4e^{4x} + t^2 \rightarrow Ae^{4x} + Bt^2 + Ct + D$

- If $g(x)$ is made up of multiple functions, you may take appropriate guesses for each to form a composite $y_p(x)$, made up of $y_{p_1}(x)$ and $y_{p_2}(x)$.
- Similarly, a $g(x)$ comprised of *multiplied* terms from above corresponds to the same guesses *multiplied*

- When $\alpha e^{\beta x}$ is multiplied by some other term in $g(x)$, you may omit the exponential term completely when finding a guess for $y_p(x)$. Once a guess is made for terms excluding $e^{\beta x}$, one can multiply it back in without needing to consider any additional coefficients. Inevitably, if one were to include a constant here, it would combine with all others and become redundant.

- Notice that, for all constants α in $g(x)$, our guess is not affected. Generally speaking, constants on the outside of $g(x)$ will have no bearing on our solving algorithm for $y_p(x)$.

For instance: If we encounter $g(x) = \pi e^{4t}\cos(5t)$, we can proceed with our guess for the last term (see the chart above), $A\cos(5t) + B\cos(5t)$, and then "tack on" on the exponential to yield $y_p(x) = e^{4t}(A\cos(5t) + B\cos(5t))$.

Once we are satisfied with our guess, which by now is full of unknown constants, we may plug in appropriate derivatives into the original ODE, since, as with $y_1(x)$ and $y_2(x)$, $y_p(x)$ must also be a solution to $L[y] = g(x)$. By collecting like terms, we will (with a lot of algebra) be able to determine all unknown constants.

Now, suppose $g(x)$ itself solves the homogeneous ODE, i.e. $L[g(x)] = 0$. In that case, a whole host of problems begin to crop up. After all, we're trying to form an equivalency between our guess and $g(x)$, and finding that $0 = g(x)$ is not so helpful. Hence, choose the lowest integer n such that x^n is *not* a solution to the ODE and multiply it by your guess. This should, if done correctly, remedy any $L[g(x)] = 0$ issues.

Finding undetermined coefficients in "guesses" of well-chosen expressions for $g(x)$ is both a highly methodical and limited process. A much stronger sledgehammer, "Variation of Parameters," is better suited for finding complicated particular solutions.

PRINCIPLE OF SUPERPOSITION

As noted above, for any two complimentary solutions of a linear, second order ODE, the following holds:

$$y_{\mathfrak{B}} = c_1 y_1 + c_2 y_2$$

This is called the "Principle of Superposition."

Reduction of Order

Suppose we're only given one solution to $L[y] = 0$. Then, by the principle of superposition, we should be able to find a relation between the known solution, y_1, and the unknown solution, y_2. The method for finding this is called the Reduction of Order, and here is the general algorithm you'll need to solve reduction of order questions:

1. Find an appropriate first solution, y_1. This is usually provided.

2. It follows from the principle of superposition that $y_2(x) = u(x)y_1(x)$ for some function $u(x)$.

3. Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, where $y = u(x) \cdot y_1(x)$ and y_1 is written in terms of x only.

4. By plugging these derivatives into $L[y] = 0$, we will yield a new ODE to solve, which, with the substitution $v = u'$, will be separable.

5. Once an expression for $u(x)$ is found, we can finally multiply this by y_1 to yield our second solution to the ODE.

Note: As in (4), any second order ODE of the form $y''(x) = f(x, y')$ may be reduced to the first order with the like substitution, $u(x) = y'(x)$. A similar substitution, $u(y) = y'(y)$ will work for second order autonomous ODEs (i.e. $y'' = f(y, y')$), though in this case we will yield $u'(y) = \frac{f(y,u)}{u}$.

Linear Independence and the Wronskian

Some of you algebraically inclined might recognize $y_{\mathcal{B}}$ as a *linear combination* of y_1 and y_2. Thus, there is a notion of *linear independence* in our solution set, where $c \cdot y_1$ may or may not be equal to y_2.

For a pair of solutions, it is easy to check for linear independence by dividing each equation to yield a (possible) constant, checking if the appropriate multiple for one term in the solution set is appropriate for other terms, or any such method to verify the existence of a scalar multiple. When comparing 3 or more solutions, it is not so easy to determine linear independence. By using the definition

$$c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) = 0 \text{ with } c_1, c_2, ..., c_n = 0$$

$$\iff y_1, y_2, ..., y_n \text{ is linearly independent}$$

we may make our problem easier with a clever application of determinate matrices.

Define the Wronskian as the following matrix:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

From our knowledge of linear algebra, we know that if $\det M = 0$ for any matrix, we have linearly dependence of its columns. This is not necessarily true in the case of functions and the Wronskian, even though differentiation is a fundamentally linear process. However, we do have the following:

$$W(y_1, y_2) \neq 0 \implies y_1 \text{ and } y_2 \text{ are linearly independent}$$

Note: Even if $W = 0$ for *some* x_0 or collection thereof, it may not be the case for *all* $x \in \mathbb{R}$ that $W = 0$. Thus, only one chosen x_i is needed to prove independence, while we would have to compute an uncountably infinite number of Wronskians to show linear dependence.

At its most basic, linear independence can be defined with the following:

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0 \quad \text{with} \quad c_1, c_2, ..., c_n = 0$$

Hence, if there exists some non-zero collection of constants such that the above inequality holds, then $[y_1, y_2, ..., y_n]$ is in fact *not* a linear independent set of solutions.

This is an essential condition for Variation of Parameters. Up next:

Variation of Parameters

In the section on non-homogeneous second order ODEs, I glossed over variation of parameters, as we didn't yet have the tools for it. Now we do, though, and thus let's consider a non-homogeneous ODE of the form

$$\boxed{L[y] = a(x)y'' + b(x)y' + c(x)y = g(x)}$$

where the functions $a, b,$ and c are *not* necessarily constants.

The first thing we'll have to do is redefine $g(x)$ as, in fact, $g(x)/a(x)$. In other words, from the $g(x)$'s perspective, y'' must have a coefficient of one. I know this sounds silly, but the conditions of the proof for variation of parameters, i.e. math magic, requires it. For further information, consult your nearest masochistic analyst.

Then, let $y_c = c_1y_1 + c_2y_2$ be the set of complimentary solutions to $L[y] = 0$, the homogeneous variant of our ODE. Just as we did in Reduction of Order, we may conjecture that y_p will be of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

By math magic, the following will define our mystery functions:

$$u_1 = -\int \frac{y_2 \cdot g(x)}{W(y_1, y_2)}dx \quad \text{and} \quad u_2 = \int \frac{y_1 \cdot g(x)}{W(y_1, y_2)}dx$$

You may disregard the arbitrary constants that will come out of the integration process, and thus yield the general solution:

$$\begin{aligned} y(x) &= \underbrace{c_1y_1 + c_2y_2}_{y_c} + \underbrace{u_1 \cdot y_1 + u_2 \cdot y_2}_{y_p} \\ &= c_1y_1 + c_2y_2 + y_p \end{aligned}$$

III N^{th} Order Equations

For n^{th} order linear ODEs, we have the familiar form

$$\boxed{y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_{n-1}(x)y'(x) + p_n(x)y(x) = g(x)}$$

where $y^{(n)}$ represents the n^{th} derivative of y. We will only be able to solve select, *linear* n^{th} order ODEs, since their size can make them difficult to work with.

CONSTANT COEFFICIENT

Homogeneous

We'll solve constant coefficient equations in the n^{th} order just as we solved them in the second order, where the following characteristic equation can be formed and its roots found:

$$r^n + ar^{n-1} + ... + yr + z = 0$$

With the roots of this equation found, we can use methods seen in Part II to form a general solution, where any factors beyond the two that are normally found in a second-order characteristic equation are put into their appropriate forms and tacked onto the existing complimentary solution.

Polynomial Factorization Review

Time to break out some high school math! As a refresher, let's consider the factorization of the n^{th} order polynomial shown above. By necessity (and by math magic), at least one factor, r_0, of the z term *will* be a root, and we can check for these *a la* plug-and-chug.

FOR EXAMPLE: if $z = 4$, then $r_0 = \pm 4 \vee \pm 2 \vee \pm 1$

Once a root is found, we can divide our polynomial by $(r - r_0)$, either creating a simple quadratic or another polynomial of the $n \geq 3^{th}$ degree, which we can thusly factor using the method described above. Additionally, if we are lucky and encounter (anywhere in the process) a polynomial of the form:

$$ar^n + br^{n-1} + cr^{n-2} + dr^{n-3} = 0 \quad \text{with} \quad \frac{a}{b} = \frac{c}{d}$$

then we can easily factor out the greatest common denominator of each "half" and simplify our work.

Non-Homogeneous

Suppose we are asked to solve a constant coefficient equation that has a real, non-zero $g(x)$ on the right-hand side. Our first step, no matter the method described below, will be to find the solution to the homogeneous variant, $L[y] = 0$, of our equation.

Undetermined Coefficients

Our method of undetermined coefficients here is nigh-identical to the methods seen above, with the same "guesses" made and tricks employed. The only real difference is difficulty, as further differentiation will be required to determine our constants.

FUNDAMENTAL SOLUTION SETS

We will revisit variation of parameters to solve some more complicated non-homogeneous ODEs, but first we'll need to take a segue into higher order Wronskians.

Linear Independence

As with second order equations, the principle of superposition can be applied to the above ODE, and thus we can represent our solution space with the following:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) = \sum_{i=1}^{n} c_i y_i(x)$$

The solution space $y(x)$ is called "a fundamental set of solutions" if the associated Wronskian proves linear independence. Of course, we'll have to expand it beyond the baby 2x2 from Part II:

$$W(y_1, y_2, ..., y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & & \\ \vdots & \vdots & & \ddots & \\ y_1^{(n)} & y_2^{(n)} & & & y_n^{(n)} \end{vmatrix}$$

Just as before, $W(y_1, y_2, ..., y_n)$ must equal 0 for *all* $x \in I$ to show linear dependence on $I \in \mathbb{R}$, whereas even if there exists *one* $x_0 \in I$ such that $W(y_1, y_2, ..., y_n)(x_0) \neq 0$, then we can assert that we have a fundamental solution set on the interval.

Abel's Identity

Up to this point, the Wronskian has been defined in terms of *known* solutions to our ODE, but thanks to our friend Abel, we are able to determine the Wronskian of a homogeneous equation using *only* information from the ODE we started with. Consider the following:

$$W(y_1, y_2, ..., y_n) = Ce^{-\int p_1(x)dx}$$

$p_1(x)$, in this case, is pulled from the following general form, which we saw above:

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_n(x)y(x) = 0$$

Abel's Identity gives an important result for homogeneous equations, being that their solution set must *always* be linearly dependent (where $C = 0$) or *always* linearly independent (where $C \neq 0$).

REVISITING VARIATION OF PARAMETERS

With all that n$^{\text{th}}$ order Wronskian knowledge under our belt, we can now begin working on variation of parameters... right? Almost. Let's defined these *special* Wronskians as follows:

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 & \cdots & y_n \\ 0 & y_2' & y_3' & \cdots & y_n' \\ 0 & y_2'' & y_3'' & & \\ \vdots & \vdots & & \ddots & \\ 1 & y_2^{(n)} & & & y_n^{(n)} \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 & \cdots & y_n \\ y_1' & 0 & y_3' & \cdots & y_n' \\ y_1'' & 0 & y_3'' & & \\ \vdots & \vdots & & \ddots & \\ y_1^{(n)} & 1 & & & y_n^{(n)} \end{vmatrix}$$

$$\ldots W_n = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & 0 \\ y_1' & y_2' & y_3' & \cdots & 0 \\ y_1'' & y_2'' & y_3'' & & \\ \vdots & \vdots & & \ddots & \\ y_1^{(n)} & y_2^{(n)} & & & 1 \end{vmatrix}$$

You'll notice that, notably, W_i has its i^{th} column swapped with the vector $\langle 0, 0, 0, ..., 1\rangle$. Putting that aside, let's consider the general solution to the ODE $L[y] = g(x)$:

$$y_{\mathfrak{B}} = y_c + y_p$$

As seen before, y_c, the "complimentary solution" is derived from $L[y] = 0$, and y_p is dependent upon $g(x)$. Suppose we have found a fundamental solution set for $L[y] = 0$. Define y_p, then, as

$$y_p = u_1 y_n + u_1 y_n + ... + u_n y_n$$

where y_i are solutions to our homogeneous equation.

Finally, we have the following formula for each u_i:

$$u_1 = \int \frac{W_1 \cdot g(x)}{W(y_1, y_2, ..., y_n)} dx \quad u_2 = \int \frac{W_2 \cdot g(x)}{W(y_1, y_2, ..., y_n)} dx$$

$$\ldots u_n = \int \frac{W_n \cdot g(x)}{W(y_1, y_2, ..., y_n)} dx$$

Equivalently, if we let W_i be equal to W with the i^{th} column instead replaced by $\langle 0, 0, 0, ..., g(x)\rangle$, then we can express u_i as $\int {W_i}/{W}$.

While functional identically, the solving process here may be much easier or much harder depending upon the problem.

EXISTENCE AND UNIQUENESS

For Linear ODEs

For linear ODEs, it's quite easy to show that a unique solution exists. Here is what to consider:

1. The initial conditions $y(x_0) = y_0,\ \ y'(x_0) = y_1,\ \ y''(x_0) = y_2, ...,\ \ y^{(n)}(x_0) = y_n$ exist and are defined for a chosen x_0
2. The functions $p_1(x), p_1(x), ..., p_n(x)$, as defined above, are continuous on the interval $I \in \mathbb{R}$

If both these conditions are met, then a *unique* solutions exists on the interval I. Of course, if $I = \mathbb{R}$, then $L[y] = g(x)$ has only one solution for all real numbers.

Extended Picard–Lindelöf Theorem

Just as we had conditions for existence and uniqueness of solutions to first order ODEs, so do we have new conditions to establish the existence and uniqueness of solutions to n^th order problems. The following mimics exactly what the Picard–Lindelöf Theorem established previously.

Let $f(x, y', y'', ..., y^{(n-1)}) = y^{(n)}$ be the general form of any n^th order ODE, with initial conditions $y(x_0) = y_0,\ \ y'(x_0) = y_1,\ \ y''(x_0) = y_2, ...,\ \ y^{(n)}(x_0) = y_n$

If $f(x, y', y'', ..., y^{(n-1)})$ is continuous at x_0 and $\frac{\partial f}{\partial y}$ is continuous for all y_i with $i \in \{0, 1, 2, ..., n\}$, **then** there exists a unique solution to the ODE for initial conditions $y(x_0) = y_i$.

IV Series Solutions

IMPORTANT DEFINITIONS

A good portion of this section will be review from Calculus 2, but nevertheless let's redefine some concepts for ourselves.

Radius of Convergence

With $A = \sum_{n=1}^{\infty} a_n(x - x_0)^n$ as our general form for power series, A is said to *converge* for a selection of ρ such that $|x - x_0| < \rho$. We can almost always use the ratio test to determine ρ and thus the convergence of A:

$$A \text{ converges} \iff |x - x_0| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Equivalently, we have

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \rho$$

Note that if $\rho = \infty$, then A converges for all $x \in \mathbb{R}$, and on the contrary if $\rho = 0$, then the series only converges for x_0.

Real Analytic

Simply put, a function $f(x)$ is *real analytic* if its Taylor series representation has a positive radius of convergence and hence converges to $f(x)$ when $|x - x_0| < \rho$. In the content to follow, almost all of the functions we'll analyze will be analytic (all elementary functions, especially, are analytic, though $|x|$ is not).

Ordinary Point

A point x_0 is said to be an *ordinary point* of $f(x)$ if the Taylor series expansion of $f(x)$ at x_0 is real analytic. Points that are not ordinary are called "singular."

If we are given an ordinary point, x_0, and two polynomials, $A(x)$ and $B(x)$, the radius of convergence of $\frac{A(x)}{B(x)}$ about x_0 is the euclidean distance between the nearest zero in the complex plane and x_0. Why? Who knows.

Regular Singular Point

Suppose that x_0 is a singular point, i.e. *not* an ordinary point. Then, if

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2\frac{R(x)}{P(x)} \text{ are analytic at } x_0$$

x_0 is called a "regular singular point." Equivalently, if $P(x)$, $Q(x)$, and $R(x)$ are polynomials, we may use another set of conditions:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \text{ is finite} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \text{ is finite}$$

If the above equations are *not* real analytic, then we have a "irregular singular point." We'll define $P(x)$, $Q(x)$, and $R(x)$ shortly.

SOLVING AN ODE USING SERIES

The Homogeneous Case

Here, we will consider the ODE

$$\boxed{L[y] = P(x)y'' + Q(x)y' + R(x)y = 0}$$

where $P(x)$, $Q(x)$, and $R(x)$ are all polynomials.

If $p(x) := \frac{Q(x)}{P(x)}$ and $q(x) := \frac{R(x)}{P(x)}$ have an ordinary point x_0, (essentially, $P(x_0) \neq 0$), we have a nice enough setup to express our solution as

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0y_1(x) + a_1y_2(x)$$

where a_0 and a_1 are arbitrary constants (recall the principle of superposition) and $y_{1,2}(x)$ are almost always power series themselves.

Note that *any* choice of x_0, so long as it is an ordinary point, can be used in our power series for $y(x)$. You'll find that fixing $x_0 = 0$ is often helpful.

Here are some statements we can make about the solution set:

- y_0 and y_1 will form a fundamental solution set, with $W(y_0, y_1) = 1$

- Let $\rho_{f(x)}$ denote the radius of convergence of the power series of $f(x)$ at x_0. Then we can form a lower bound for our solution:

$$\rho_{y(x)} \geq \min\{\rho_{p(x)}, \rho_{q(x)}\}$$

In order to solve our ODE, we will need to differentiate our series solution:

$$y'(x) = \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1} \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

However, since $na_n(x-x_0)^{n-1}\big|_{n=0}$ and $n(n-1)a_n(x-x_0)^{n-2}\big|_{n=0,1}$ are both equal to 0, we can equivalently write

$$y'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$$

or even...

$$y'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x - x_0)^n \quad \text{and} \quad y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n$$

by re-indexing the altered sum.

By utilizing these transformations, we can reduce our work when it comes to reducing, factoring, and combining terms of $y(x)$ (the problem-solving process is especially painful when using series solutions).

In the event that $P(x)$, $Q(x)$, and $R(x)$ are not polynomials, we can rearrange our ODE to yield $L[y] = y'' + p(x)y' + q(x)y = 0$ and thus solve.

Non-Homogeneous

When $L[y] = g(x)$, our solving process is similar but nonetheless more dreadful than before. In the end, our solution should be in the following form:

$$y(x) = a_0 y_1(x) + a_1 y_2(x) + y_p(x)$$

The "trick," in this case, will be to separate terms of x^n in y_p in order to solve a system of equations. If this is not possible, then we are out of luck when it comes to using the simple stuff.

Euler Equations

We've gone long enough without defining a new ODE form, so let's do that. A Euler equation (in its most general form) looks like

$$\boxed{L[y] = \sum_{i=0}^{n} = a_i(x)x^i y^{(i)}}$$

To solve, we'll first consider the homogeneous, constant-coefficient case where $n = 2$, i.e. $L[y] = ax^2y'' + bxy' + cy$. Assuming that x^r satisfies this equation, we can form the following *indicial equation* to help us find r:

$$ar(r-1) + br + c = 0$$

From here, there are three main cases to consider (just like the characteristic polynomial for second order equations).

Real Roots

Having factored our indicial equation, if we have two distinct, real roots, we can express the general solution as

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}$$

with $c_1, c_2 \in \mathbb{R}$. Since $r_1 \neq r_2$, our solution set is automatically linearly independent.

Complex Roots

If we have complex roots, in the form $r = \alpha \pm \beta i$, then our general solution is

$$\begin{aligned} y(x) &= c_1|x|^\alpha \cos(\beta \ln|x|) + c_2|x|^\alpha \sin(\beta \ln|x|) \\ &= |x|^\alpha \left(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\right) \end{aligned}$$

Repeated Roots

With two repeated roots, $r = r_1 = r_2$, the general solution looks like

$$y(x) = |x|^r(c_1 + c_2 \ln|x|)$$

Frobenius' Method

This next section will be especially messy, so I suggest referencing chapter 5.5 in the 12$^{\text{th}}$ edition of Wiley's *Elementary Differential Equations* for a more rigorous presentation. Suppose we have an ODE of the form $P(x)y'' + Q(x)y' + R(x)y = 0$, but we'd like to form our solution around a regular singular point as opposed to an ordinary point. In this case, the following algorithm, "Frobenius' Method," will help us solve for $y(x)$:

1. Rearrange the ODE such that $x^2y'' + xp(x)y' + q(x)y = 0$

2. Find a regular singular point, x_0, such that

$$\lim_{x\to x_0}(x - x_0)p(x) = p_0 \quad \text{and} \quad \lim_{x\to x_0}(x - x_0)^2 q(x) = q_0$$

   are finite. It is useful to consider p_n as the coefficient contained in the n^{th} degree of $x \cdot p(x)$'s Taylor expansion. The same holds for $x^2 \cdot q(x)$.

3. Derive the associated Euler equation $x^2y'' + p_0xy' + q_0y = 0$ using the values found in the last step.

4. Use the indicial equation, $F(r) = 0$, to find r_1 and r_2

5. We will have

$$\boxed{F(r+n)a_n + \sum_{k=0}^{n-1} a_k \left[(r+k)p_{n-k} + q_{n-k}\right] = 0}$$

This is quite daunting at first. Think of the summation as considering terms $p_{1,2,...}$ and $q_{1,2,...}$ which are non-zero. We will not have to consider r for the moment, and thus find a general recurrence relation with $a_n = \{\text{something}\} \cdot a_{n-1}$

6. Set $r = r_1$ and use it to find a general formula for a_n (it will contain a_0 as a constant, which we can remove)

7. With a_n, we can substitute in equation (1) to get a formula for $y_1(x)$. This can be rewritten in the form

$$y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_{n\{r_1\}} x^n\right)$$

8. An expression for $y_2(x)$ can similarly be derived using $r = r_2$ and repeating steps 5-6. Our general solution will be of the familiar form $y(x) = c_1 y_1(x) + c_2 y_2(x)$[3].

9. And we're done!

However, I lied slightly in step (8). Our solution will have to be tweaked if $r_1 = r_2$, i.e. if our indicial equation from step (6) has repeated roots. In this case, $y_1(x)$ may remain in its current form, but $y_2(x)$ should be altered as such:

$$y_2(x) = y_1(x)\ln(x) + x^{r_1}\sum_{n=1}^{\infty} b_n x^n$$

with $b_n = \frac{\partial}{\partial r}\left[a_{n\{r_1\}}\right]$. If b_n cannot be found using partial differentiation, it is possible (though awful) to find first and second derivatives of $y_2(x)$ itself and plug it into the original ODE.

[3]This is why it was OK to remove a_0 as we did in step (5)

V Laplace Transforms

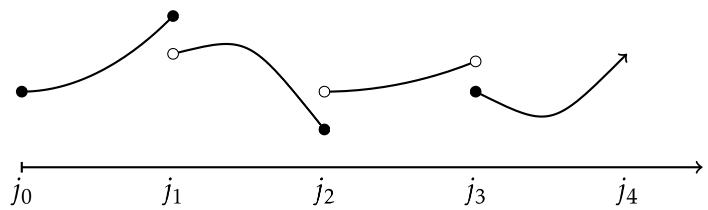
DEFINITION

Laplace transforms are, in effect, fancy integrals, and they are applicable to *continuously piecewise* functions, or piecewise functions whose open subintervals are themselves continuous. Of course, any normally continuous function satisfies this condition.

The Laplace transform is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Note that, for piecewise continuous functions on an interval I, we can divide our interval into N subintervals, with $j_1, j_2, ..., j_N$ denoting each dividing point.



Thus, we can also write our Laplace transform as

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{N} \int_{j_n}^{j_{n+1}} e^{-st} f(t) dt$$

Laplace transforms are also linear transformations! Like all linear transformations, we can rearrange constants and terms as such:

$$\begin{aligned}
\mathcal{L}\{Af(x) + Bg(x) + h(x)\} &= \mathcal{L}\{Af(x) + Bg(x)\} + H(x) \\
&= \mathcal{L}\{Af(x)\} + \mathcal{L}\{Bg(x)\} + H(x) \\
&= AF(x) + BG(x) + H(x)
\end{aligned}$$

In addition to being piecewise continuous, a function must be of *exponential order* in order for its Laplace transform to exist. In particular, a function $f(x)$ is said to be of exponential order if

$$|f(x)| \leq Me^{ct} \quad \forall t \geq T \quad \text{for positive constants } M, T, \text{and } c$$

For instance, $e^{\alpha t}$ meets this condition with $c = \alpha$, $M = 1$, and T being arbitrary.

Putting this all together, we have the following existence condition:

If $|f(x)| \leq Me^{ct} \quad \forall t \geq T$ with $M, T, c > 0$ and $f(x)$ is piecewise continuous on $t \in [0, \infty)$, **then** its Laplace transform exists.

SOLVING IVPS USING LAPLACE TRANSFORMS

Ordinary Functions

All of this preamble about the Laplace transform boils down to the following equation:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

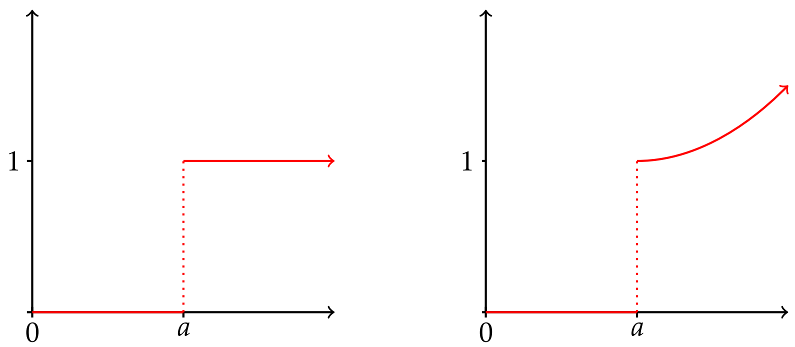
Here, we have the multipliers s decreasing in exponential magnitude while the derivatives of our "initial conditions," $f(0)$, $f'(0)$, $f''(0)$, etc. are *increasing* in magnitude. Once $s^n \to s^0$, we can stop. From here on out, solving IVPs this way will just be a matter of moving certain expressions in and out of its Laplace transformation.

Heaviside Functions

Define the Heaviside function, or the unit step function, as the following:

$$\mathcal{U}_a = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t \geq a \end{cases}$$

Visually, this acts as an "off-on" switch:

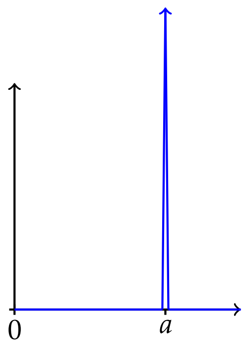


$\mathcal{U}_a$ $\qquad$ $\mathcal{U}_a \cdot f(t)$

We can easily form other Heaviside-like constructions by multiplying $\mathcal{U}_a$ by any otherwise continuous function (shown on the right). It's Laplace transform is given on the next page.

The Dirac Delta Function

The Dirac delta function, denoted $\delta(t-a)$, can be thought of as a short *impulse* of energy, with infinite magnitude, infinitesimal duration, and (curiously) finite area. Here's an exaggerated picture:



$\delta(t-a)$

It's Laplace transform is also listed in the table below.

Table of Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} f(t)$	$F(s-a)$ or the First Translation Theorem
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\mathcal{U}_a$	$\frac{e^{-as}}{s}$
$\mathcal{U}_a f(t-a)$	$e^{-as}F(s)$ or the Second Translation Theorem
$\delta(t-a)$	e^{-as}

CONVOLUTIONS

Define a *conovlution* of two functions as

$$(f \circledast g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

We won't be concerned too much with the geometric intuition of convolutions, but they nevertheless are useful in solving IVPs. In particular, we have this relation:

$$\mathscr{L}\{(f \circledast g)(t)\} = \mathscr{L}\{f(t)\} \cdot \mathscr{L}\{g(t)\}$$

From here, it is possible to derive a solution for $y(t)$ using convolutions. You'll avoid all the nasty partial fractions in doing so, but at the same time introduce *many* messy integrals (so, generally, it's not worth it). The following method utilizes convolutions in a far more practical way. Let's take a non-homogeneous ODE, compute its Laplace transform, and rearrange it as such:

$$\boxed{L[y](t) = f(t) \to P(s)Y(s) - Q(s) = F(s)}$$

Then we have:

$$y(t) = (f \circledast g)(t) + \mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$

Where $g(t)$, the "Green's function," has the following properties:

$$L[g](t) = \delta(t) \quad \text{with} \quad g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0$$

Thankfully, there's a straightforward expression for $g(t)$:

$$\mathscr{L}\{g(t)\} = \frac{1}{P(s)}$$

Cheers!