Introduction

**Definition 0.1** (Riemann 1854). Let \([a, b]\) be a closed bounded interval, \(f : [a, b] \to \mathbb{R}\) bounded function. We say \(f\) is **Riemann integrable** if

\[
\int_a^b f := \sup \left\{ \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(x_i - x_{i-1}) : a = x_0 < x_1 < \cdots < x_n = b \right\}
\]

\[
= \int_a^b f := \inf \left\{ \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x_i - x_{i-1}) : a = x_0 < x_1 < \cdots < x_n = b \right\}
\]

We then denote \(\int_a^b f = \int_a^b f(x) \, dx := \int_a^b f = \int_a^b f\).

**Theorem 0.1.** Every continuous function \(f : [a, b] \to \mathbb{R}\) is Riemann integrable.

**Remark.** \(f : x \in [0, 1] \mapsto \begin{cases} 
1 & x \in \mathbb{Q} \\
0 & x \notin \mathbb{Q} 
\end{cases}\) is not Riemann integrable.

1 Measure Theory

**Definition 1.1.** 1. Let rectangle \(R\) be \((a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq R \subseteq [a_1, b_1] \times \cdots \times [a_d, b_d]\), where \(-\infty < a_i \leq b_i < \infty \forall 1 \leq i \leq d\). We call **volume** of \(R\) and denote \(\text{vol}(R)\) the number \(\text{vol}(R) := \prod_{i=1}^{d} (b_i - a_i)\). We say that \(R\) is a **cube** if \(b_1 - a_1 = \cdots = b_d - a_d\).

2. For every set \(A \subseteq \mathbb{R}^d\) we call the **exterior measure** of \(A\) and denote \(m_*(A)\) the number

\[
m_* (A) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(Q_k) : Q_k \text{ closed cubes}, A \subseteq \bigcup_{k=1}^{\infty} Q_k \right\} \in [0, \infty]
\]

**Remark.**

\[
\left\{ \sum_{k=1}^{\infty} \text{vol}(Q_k) : Q_k \text{ closed cubes}, A \subseteq \bigcup_{k=1}^{\infty} Q_k \right\} \neq \emptyset \implies A \subseteq \bigcup_{n=1}^{\infty} [-n, n]^d = \mathbb{R}^d
\]
Remark.

\[ m_*(A) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(Q_k) : Q_k \text{ open cubes, } A \subseteq \bigcup_{k=1}^{\infty} Q_k \right\} \]

\[ = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(Q_k) : Q_k \text{ rectangles, } A \subseteq \bigcup_{k=1}^{\infty} Q_k \right\} \]

**Proposition 1.1.** If \( A \subseteq \mathbb{R}^d \) is countable then \( m_*(A) = 0 \)

**Proposition 1.2** (monotonicity). If \( A \subseteq B \subseteq \mathbb{R}^d \) then \( m_*(A) \leq m_*(B) \)

**Proposition 1.3.** If \( O \subseteq \mathbb{R}^d \) is open then it can be written as \( O = \bigcup_{k=1}^{\infty} \overline{Q}_k \) where \( Q_k \) are disjoint, open cubes (\( \overline{Q}_k \) is the closure of \( Q_k \)).

**Proposition 1.4.** If \( R \subseteq \mathbb{R}^d \) is a rectangle then \( m_*(R) = \text{vol}(R) \).

**Proposition 1.5.** If \( A \subseteq \mathbb{R}^d \) then \( m_*(A) = \inf \{ m_*(O) : O \text{ open set, } A \subseteq O \} \).

**Proposition 1.6.** Let \((A_k)_{k \in \mathbb{N}}\) be a sequence of sets in \( \mathbb{R}^d \) (not necessarily disjoint). Then \( m_*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} m_*(A_k) \).

**Proposition 1.7.** Let \( A_1, A_2 \subseteq \mathbb{R}^d \) be such that \( d(A_1, A_2) > 0 \) i.e. \( \inf \{|x-y| : x \in A_1, y \in A_2\} > 0 \). Then \( m_*(A_1 \cup A_2) = m_*(A_1) + m_*(A_2) \)

**Definition 1.2.** A set \( A \subseteq \mathbb{R}^d \) is said to be (Lebesgue)-measurable if for every \( \epsilon > 0 \), there exists \( O_\epsilon \) open such that \( A \subseteq O_\epsilon \) and \( m_*(O_\epsilon \setminus A) < \epsilon \). We then denote \( m(A) = m_*(A) \) the (Lebesgue)-measure of \( A \).

**Proposition 1.8.**  
1. If \( m_*(A) = 0 \) then \( A \) is measurable.

2. A countable union of measurable sets is measurable.

3. Open sets and closed sets are measurable.

4. If \( A \) is measurable then \( \mathbb{R}^d \setminus A =: A^c \) is measurable.

5. A countable intersection of measurable sets is measurable.
**Theorem 1.9** (countable additivity). Let \((A_k)_{k \in \mathbb{N}}\) be measurable and disjoint. Then

\[
m \left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} m(A_k)\]

**Remark.** In particular, if \(A \subseteq B \subseteq \mathbb{R}^d\) are measurable then \(m(B) = m(A) + m(B \setminus A)\).

**Proposition 1.10** (continuity of measure). Let \((A_k)_{k \in \mathbb{N}}\) be measurable.

1. If \(A_k \subseteq A_{k+1} \forall k \in \mathbb{N}\) then \(m(\bigcup_{k=1}^{\infty} A_k) = \lim_{k \to \infty} m(A_k)\).

2. If \(A_k \supseteq A_{k+1} \forall k \in \mathbb{N}\) and \(m(A_1) < \infty\) then \(m(\bigcap_{k=1}^{\infty} A_k) = \lim_{k \to \infty} m(A_k)\).

**Remark.** \(m(A_1) < \infty\) is necessary: \(m(\bigcap_{k=1}^{\infty} [k, \infty)) = m(\emptyset) = 0\) while \(m([k, \infty)) = \infty \forall k \in \mathbb{N}\).

**Theorem 1.11** (outer and inner approximations of measurable sets). Let \(A \subseteq \mathbb{R}^d\). Then the following are equivalent:

1. \(A\) is measurable;

2. There exists a \(G_\delta\) set \(G\) (a \(G_\delta\) set is a countable intersection of open sets) and a set \(N\) of measure 0 such that \(A = G \setminus N\);

3. For every \(\varepsilon > 0\), there exists \(F_\varepsilon\) closed such that \(F_\varepsilon \subseteq A\) and \(m_*(A \setminus F_\varepsilon) < \varepsilon\);

4. There exists an \(F_\sigma\) set \(F\) (an \(F_\sigma\) set is a countable union of closed sets) and a set \(N\) of measure 0 such that \(A = F \cup N\).

**Counterexamples**

Are all subsets of \(R^d\) measurable?

**Theorem 1.12.** If \(A \subseteq \mathbb{R}^d\) is such that \(m_*(A) > 0\) then there exists \(B \subseteq A\) non-measurable.

Are all subsets of measure 0 in \(R\) countable?

**Definition 1.3.** We call Cantor set the set \(C := \bigcap_{k=1}^{\infty} C_k\) where \(C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]\) and \(\forall k \geq 2, C_k := \bigcup_{j=1}^{2^k} I_{j,k}\) where \(\forall j \in \{1, \ldots, 2^{k-1}\}, I_{2j-1,k}, I_{2j,k}\) are the first and last thirds of \(I_{j,k-1}\).

**Theorem 1.13.** \(C\) is closed and uncountable. \(m(C) = 0\).
Are all measurable sets Borel?

**Definition 1.4.** A collection $\Omega$ of subsets of $\mathbb{R}^d$ is called a $\sigma$-algebra if the following conditions are satisfied:

1. $\mathbb{R}^d \in \Omega$;
2. $\forall A, B \in \Omega : A \setminus B \in \Omega$;
3. $\forall (A_k)_{k \in \mathbb{N}} \subseteq \Omega : \bigcup_{k=1}^{\infty} A_k \in \Omega$.

**Proposition 1.14.** Any intersection of $\sigma$-algebras is a $\sigma$-algebra.

**Definition 1.5.** The intersection of all the $\sigma$-algebras containing the open sets is called the **Borel $\sigma$-algebra** and its elements the **Borel sets**.

**Remark.** In particular, Borel sets are measurable.

**Proposition 1.15.** There exists a subset of the Cantor set which is measurable but not Borel.

**Definition 1.6.** We call **Cantor-Lebesgue function** (or **Cantor staircase function**) the function

$$
\varphi : [0, 1] \to [0, 1],
\varphi(x) = \frac{i}{2^k} \text{ if } x \in J_{k,i} \text{ where } J_{k,i} \text{ is the } i\text{-th interval of } [0, 1] \setminus C_k, k \geq 1, i \in \{1, \ldots, 2^k - 1\},
\varphi(0) = 0, \varphi(x) = \sup\{\varphi(y) : y \in [0, x) \setminus C\} \text{ if } x \in (0, 1] \cap C
$$

**Remark.** $\varphi(1) = 1$.

**Proposition 1.16.** $\varphi : [0, 1] \to [0, 1]$ is increasing, continuous and surjective.

**Proposition 1.17.** If $D \subseteq \mathbb{R}$ is not Borel, then $D \times \{0\}^{d-1} \subseteq \mathbb{R}^d$ is not Borel.

## 2 Lebesgue Measurable Function

**Remark.** We denote $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

**Proposition 2.1.** Let $A \subseteq \mathbb{R}^d$ be measurable, $f : A \to \bar{\mathbb{R}}$. Then the following are equivalent:

1. $\forall c \in \mathbb{R} : f^{-1}((c, +\infty])$ is measurable
2. \( \forall c \in \mathbb{R} : f^{-1}([c, +\infty)) \) is measurable

3. \( \forall c \in \mathbb{R} : f^{-1}([-\infty, c)) \) is measurable

4. \( \forall c \in \mathbb{R} : f^{-1}([-\infty, c]) \) is measurable

**Definition 2.1.** When these are satisfied, we say \( f \) is (Lebesgue) measurable.

**Proposition 2.2.** Let \( A \subseteq \mathbb{R}^d \) and \( (A_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^d \) be measurable sets such that the sets \( (A_k)_{k \in \mathbb{N}} \) are disjoint and \( \bigcup_{k=1}^{\infty} A_k = A \). Let \( f : A \to \mathbb{R} \) be a function. If \( f|_{A_k} \) is measurable for all \( k \in \mathbb{N} \) then \( f \) is measurable.

**Proposition 2.3.** Let \( A \subseteq \mathbb{R}^d \) measurable.

1. \( \forall B \subseteq A \) measurable, \( \forall f : A \to \mathbb{R} \) measurable, \( f|_{B} \) is measurable;

2. \( \forall B \subseteq \mathbb{R} \) Borel, \( \forall f : B \to \mathbb{R} \) continuous, \( \forall g : A \to B \) measurable, then \( f \circ g \) is measurable;

3. \( \forall f : A \to \mathbb{R}, \forall g : A \to \mathbb{R} \) both measurable, \( f + g \) is measurable;

4. \( \forall f : A \to [0, \infty] \) measurable, \( \forall k \in \mathbb{N}, f^k \) is measurable;

5. \( \forall f, g : A \to \mathbb{R} \) measurable, \( f \cdot g \) is measurable;

6. \( \forall f, g : A \to \mathbb{R} \) measurable, \( \max(f, g), \min(f, g) \) is measurable.

**Proposition 2.4.** Let \( A \subseteq \mathbb{R}^d \) be measurable, let \( f : A \to \mathbb{R} \) measurable. Then for every Borel set \( B \subseteq \mathbb{R} \), \( f^{-1}(B) \) is measurable.

**Remark.** \( \exists D \subseteq \mathbb{R} \) measurable, \( f \) measurable (even continuous) such that \( f^{-1}(D) \) is not measurable.

**Proposition 2.5.** Let \( A \subseteq \mathbb{R}^d \) measurable, \( f : A \to \mathbb{R} \) continuous, then \( f \) is measurable.

**Definition 2.2.** Let \( A \subseteq \mathbb{R}^d \), \( P(x) \) a statement depending on \( x \in A \). We say \( P(x) \) is true for almost every \( x \in A \) (or a.e. \( x \in A \)) if \( m_a(\{x \in A : P(x) \text{ is false}\}) = 0 \).

**Proposition 2.6.** If \( (P_k(x))_{k \in \mathbb{N}} \) is a countable collection of statements depending on \( x \in A \), then

\[
[\forall k \in \mathbb{N} : \text{for a.e. } x \in A, P_k(x) \text{ is true}] \Leftrightarrow [\text{for a.e. } x \in A, \forall k \in \mathbb{N} : P_k(x) \text{ is true}]
\]
Proposition 2.7. Let $f, g : A \to \mathbb{R}$ be such that $f = g$ a.e. in $A$. Then $f$ measurable if and only if $g$ measurable.

Proposition 2.8. Let $A \subseteq \mathbb{R}^d$ and $(A_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^d$ be measurable sets such that the sets $(A_k)_{k \in \mathbb{N}}$ disjoint and $\bigcup_{k=1}^{\infty} A_k = A$. Let $f : A \to \mathbb{R}$ be a function. If $f|_{A_k}$ is measurable for all $k \in \mathbb{N}$, then $f$ is measurable.

Proposition 2.9. Let $A \subseteq \mathbb{R}^d$ be measurable

1. $\forall B \subseteq A$ measurable, $\forall f : A \to \mathbb{R}$ measurable, $f|_B$ is measurable.

2. $\forall B \subseteq \mathbb{R}$ Borel, $\forall f : B \to \mathbb{R}$ continuous, $\forall g : A \to B$ measurable, $f \circ g$ is measurable.

3. $\forall f : A \to \mathbb{R}$ measurable, $\forall g : A \to \mathbb{R}$ measurable, $f + g$ is measurable.

4. $\forall f : A \to \mathbb{R}$ measurable, $\forall k \in \mathbb{N}$, $f^k$ is measurable.

5. $\forall f, g : A \to \mathbb{R}$, $f \cdot g$ is measurable.

Remark. $\exists f, g$ measurable such that $f \circ g$ is not measurable.

Proposition 2.10. Let $(f_n)_{n \in \mathbb{N}}, f_n : A \to \mathbb{R}$ be measurable functions converging pointwise a.e. in $A$ to a function $f : A \to \mathbb{R}$ i.e. $\lim_{n \to \infty} f_n(x) = f(x)$ for a.e. $x \in A$. Then $f$ is measurable.

Proposition 2.11. Let $(f_n)_{n \in \mathbb{N}}, f_n : A \to \mathbb{R}$ be measurable functions. Then

$$x \mapsto \inf_{n \in \mathbb{N}} f_n(x), x \mapsto \sup_{n \in \mathbb{N}} f_n(x), x \mapsto \liminf_{n \to \infty} f_n(x), x \mapsto \limsup_{n \to \infty} f_n(x)$$

are all measurable.

Definition 2.3. We call simple function a measurable function $\varphi : A \to \mathbb{R}$ such that $\varphi(A)$ is finite and $\varphi$ has finite support i.e. $m(\{x \in A : \varphi(x) \neq 0\}) < \infty$.

Remark. In particular, any simple function $\varphi$ can be written as

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$$

where $n \geq 0$, $c_1, \ldots, c_n \in \mathbb{R} \setminus \{0\}$ distinct (such that $\varphi(A) \setminus \{0\} = \{c_1, \ldots, c_n\}$) and $A_1, \ldots, A_n \subseteq A$ measurable, disjoint and with finite measure ($A_i = \varphi^{-1}(\{c_i\})$).
Definition 2.4. We say that $\sum_{i=1}^{n} c_i \chi_{A_i}$ is the \textit{canonical form} of the simple function $\varphi$. We say that $\sum_{i=1}^{n} c_i \chi_{A_i}$ is a \textit{step function} if the $A_i$ are rectangles.

Theorem 2.12 (Simple Approximation Lemma). Let $f : A \to \mathbb{R}, m(A) < \infty$ be measurable and bounded i.e. $\exists M > 0 \forall x \in A : |f(x)| < M$. Then $\forall \varepsilon > 0 \exists \varphi_{\varepsilon}, \psi : A \to \mathbb{R}$ simple functions such that

$$\varphi_{\varepsilon} \leq f \leq \psi_{\varepsilon} < \varphi_{\varepsilon} + \varepsilon$$

Theorem 2.13 (Simple Approximation Theorem). Let $f : A \to \mathbb{R}$ be measurable. Then there exist $(\varphi_n)_{n \in \mathbb{N}}$ simple functions such that

1. $(\varphi_n)_{n \in \mathbb{N}}$ converges pointwise to $f$ on $A$ i.e. $\lim_{n \to \infty} \varphi_n(x) = f(x) \forall x \in A$, and
2. $|\varphi_n| \leq |\varphi_{n+1}| \leq |f|$ on $A \forall n \in \mathbb{N}$.

If $f \geq 0$ then we have moreover that $\varphi_n \geq 0$ and increasing in $n$.

Theorem 2.14 (Egorov). Let $A \subseteq \mathbb{R}^d$ be measurable, $m(A) < \infty$, $(f_n)_{n \in \mathbb{N}}, f_n : A \to \mathbb{R}$ measurable converging pointwise to $f : A \to \mathbb{R}$. Then $\forall \varepsilon > 0 \exists F_\varepsilon \subseteq A$ closed such that $(f_n)_{n \in \mathbb{N}}$ converges to $f$ uniformly in $F_\varepsilon$ i.e.

$$\sup_{F_\varepsilon} |f_n - f| \xrightarrow{n \to \infty} 0$$

and $m(A \setminus F_\varepsilon) < \varepsilon$

Remark. This result does not hold in general when $m(A) = \infty$, e.g. $f_n(x) = \frac{x}{n}$ on $A = \mathbb{R}$.

Theorem 2.15 (Lusin). Let $f : A \to \mathbb{R}$ be measurable. Then $\forall \varepsilon > 0 \exists F_\varepsilon \subseteq A$ closed such that $f|_{F_\varepsilon}$ is continuous on $F_\varepsilon$ and $m(A \setminus F_\varepsilon) < \varepsilon$.

Remark. Recall that “$f|_F$ continuous on $F$” $\neq$ “$f$ continuous on $F$”: $\chi_{\mathbb{Q}}$ is not continuous at any point in $\mathbb{R}$ but $\chi_{\mathbb{Q}}|_{\mathbb{R} \setminus \mathbb{Q}} = 0$ is continuous on $\mathbb{R} \setminus \mathbb{Q}$.
3 Lebesgue Integration

Case of a simple function

Definition 3.1. Let \( \varphi : A \to \mathbb{R} \) be a simple function and \( \varphi = \sum_{k=1}^{n} c_k \chi_{A_k} \) be its canonical form. We define the (Lebesgue) integral of \( \varphi \) over \( A \) by

\[
\int_{A} \varphi = \int_{A} \varphi(x) \, dx = \sum_{k=1}^{n} c_k m(A_k)
\]

For any \( B \subseteq A \) measurable, we define \( \int_{B} f = \int_{A} f \chi_{B} \).

Proposition 3.1 (independence of the representation). Let \( n \in \mathbb{N} \), \( c_1, \ldots, c_n \in \mathbb{R} \) and \( A_1, \ldots, A_n \subseteq A \) be measurable, disjoint, \( m(A_k) < \infty \). Then

\[
\int_{A} \sum_{k=1}^{n} c_k \chi_{A_k} = \sum_{k=1}^{n} c_k m(A_k)
\]

Proposition 3.2. Let \( \varphi, \psi : A \to \mathbb{R} \) be simple. Then

1. \( \forall \alpha, \beta \in \mathbb{R} \), \( \alpha \varphi + \beta \psi \) simple and \( \int_{A} (\alpha \varphi + \beta \psi) = \alpha \int_{A} \varphi + \beta \int_{A} \psi \).

2. \( \forall B_1, B_2 \subseteq A \) measurable disjoint, \( \int_{B_1 \cup B_2} \varphi = \int_{B_1} \varphi + \int_{B_2} \varphi \).

3. If \( \varphi \leq \psi \) on \( A \) then \( \int_{A} \varphi \leq \int_{A} \psi \).

4. \( |\varphi| \) is simple and \( |\int_{A} \varphi| \leq \int_{A} |\varphi| \).

Case of a bounded measurable function with finite support

Definition 3.2. We denote \( \text{supp}(f) \) and call support of a measurable function \( f : A \to \mathbb{R} \) the set

\[
\text{supp}(f) = \{ x \in A : f(x) \neq 0 \}
\]

If \( \text{supp}(f) \subseteq E \subseteq A \), then we say \( f \) is supported in \( E \). If \( m(\text{supp}(f)) < \infty \), we say that \( f \) has finite support.

Proposition 3.3. Let \( f : A \to \mathbb{R} \) be bounded, measurable and with finite support. Let \( (\varphi_n)_{n \in \mathbb{N}} \) be simple functions in \( A \) such that
1. \( \exists E \subseteq A \) measurable such that \( m(E) < \infty \) and \( \text{supp}(\varphi_n) \subseteq E \) for all \( n \in \mathbb{N} \),

2. \( \exists M > 0 \) such that \( \forall n \in \mathbb{N}, |\varphi_n| \leq M \) in \( A \), and

3. \( \lim_{n \to \infty} \varphi_n(x) = f(x) \) for a.e. \( x \in A \) (a.e. pointwise convergence).

Then \( \lim_{n \to \infty} \int_A \varphi_n \) exists and does not depend on the choice of \( (\varphi_n)_{n \in \mathbb{N}} \) satisfying the above.

**Remark.** Such \( (\varphi_n) \) exists by the Simple Approximation Lemma.

**Definition 3.3.** Given the above proposition, we then call integral of \( f \) over \( A \) the number \( \int_A f = \lim_{n \to \infty} \int_A \varphi_n \). For every \( B \subseteq A \) measurable, we define \( \int_B f = \int_A f \chi_B \).

**Remark.** If \( f = 0 \) a.e. in \( A \) then \( \int_A f = 0 \).

**Proposition 3.4.** Let \( f, g : A \to \mathbb{R} \) be bounded, measurable and with finite support. Then

1. \( \forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g \) is bounded, measurable and with finite support and \( \int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g \).

2. \( \forall B_1, B_2 \subseteq A \) measurable disjoint, \( \int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f \).

3. If \( f \leq g \) on \( A \) then \( \int_A f \leq \int_A g \).

4. \( |f| \) is bounded, measurable and with finite support and \( |\int_A f| \leq \int_A |f| \).

**Theorem 3.5** (Bounded Convergence Theorem). Let \( (f_n)_{n \in \mathbb{N}}, f : A \to \mathbb{R} \) be a sequence of measurable function such that

1. \( \exists E \subseteq A \) measurable such that \( m(E) < \infty \) and \( \text{supp}(f_n) \subseteq E \) for all \( n \in \mathbb{N} \),

2. \( \exists M > 0 \) such that \( \forall n \in \mathbb{N}, |f_n| \leq M \) in \( A \), and

3. \( \exists f : A \to \mathbb{R} \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) for a.e. \( x \in A \).

Then \( f \) is bounded, measurable, with finite support and \( \lim_{n \to \infty} \int_A f_n = \int_A f \).

**Remark.** \( \int_0^1 n\chi_{[0, \frac{1}{n}]}(x) \, dx = 1 \) but \( n\chi_{[0, \frac{1}{n}]}(x) \xrightarrow{n \to \infty} 0 \forall x \in (0, 1] \) i.e. for a.e. \( x \in [0, 1] \).

**Theorem 3.6.** If \( A = [a,b], a < b \in \mathbb{R} \) then every bounded function \( f : A \to \mathbb{R} \) that is Riemann integrable is measurable and its Riemann \( \int_A f \) is equal to its Lebesgue integral \( \int_A f \).
Case of a nonnegative measurable function

Definition 3.4. Let \( f : A \to [0, \infty] \) be measurable. Then we define the integral of \( f \) over \( A \) as

\[
\int_A f = \sup \left\{ \int_A h : h : A \to [0, \infty) \text{ bounded, measurable, with finite support, } h \leq f \text{ on } A \right\}
\]

For every \( B \subseteq A \), we define \( \int_B f = \int_A \tilde{f} \) where \( \tilde{f}(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases} \) \( (\tilde{f} = \chi_B f \text{ if } f < \infty) \).

We say that \( f \) is integral over \( B \) if \( \int_B f < \infty \).

Proposition 3.7. Let \( f, g : A \to [0, \infty] \) be measurable. Then

1. \( \forall \alpha, \beta \geq 0, \alpha f + \beta g \) is nonnegative measurable and \( \int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g \).

2. \( \forall B_1, B_2 \subseteq A \) measurable disjoint, \( \int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f \).

3. If \( f \leq g \) on \( A \) then \( \int_A f \leq \int_A g \). Moreover, if \( f = g \) a.e. on \( A \) then \( \int_A f = \int_A g \). In particular, if \( f = 0 \) a.e. on \( A \) then \( \int_A f = 0 \).

Remark. For every \( A \subseteq \mathbb{R}^d \) measurable, \( \int_{\mathbb{R}^d} \chi_A = m(A) \).

Theorem 3.8 (Chebyshev’s Inequality). Let \( f : A \to [0, \infty] \) be measurable. Then

\[
\forall c > 0 : m(f^{-1}([0, +\infty])) \leq \frac{1}{c} \int_A f
\]

Corollary 3.9. Let \( f : A \to [0, \infty] \) be measurable. Then \( \int_A f = 0 \Leftrightarrow f = 0 \text{ a.e. in } A \).

Corollary 3.10. Let \( f : A \to [0, \infty] \) be measurable. If \( f \) is integrable then \( f < \infty \) a.e. in \( A \).

Theorem 3.11 (Fatou’s Lemma). Let \( (f_n)_{n \in \mathbb{N}} \) be measurable nonnegative on \( A \subseteq \mathbb{R}^d \). Then

\[
\int_A \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int_A f_n
\]

Remark. There is not equality since

\[
\int_\mathbb{R} n \chi_{(0, \frac{1}{n})} = 1 > \int_\mathbb{R} \lim_{n \to \infty} (n \chi_{(0, \frac{1}{n})}) = \int_\mathbb{R} 0 = 0
\]
Theorem 3.12 (Monotone Convergence Theorem). Let \((f_n)_{n \in \mathbb{N}}\) be measurable, nonnegative functions increasing in \(n\) (i.e. \(f_{n+1} \geq f_n\) on \(A\)). Then \(\lim_{n \to \infty} \int_A f_n = \int_A \lim_{n \to \infty} f_n\).

Corollary 3.13. Let \((u_n)_{n \in \mathbb{N}}\) be measurable, nonnegative function. Then \(\sum_{n=1}^{\infty} \int_A u_n = \int_A \sum_{n=1}^{\infty} u_n\).

Case of a sign-changing function

Definition 3.5. Let \(f : A \to \mathbb{R}\) be measurable. We say that \(f\) is integrable if \(f_+ = \max(f, 0)\) and \(f_- = \max(-f, 0)\) are integrable. We then call integral of \(f\) over \(A\) the number \(\int_A f = \int_A f_+ - \int_A f_-\). For every \(B \subseteq A\), we denote \(\int_B f = \int_B f_+ - \int_B f_-\).

Proposition 3.14. \(f\) integrable \(\Leftrightarrow |f|\) integrable.

Remark. If \(f, g : A \to \mathbb{R}\) then

\[
\begin{cases}
  f + g \text{ is not defined on } N = \{x \in A : f(x) = -g(x) = \pm \infty\} \\
  fg \text{ is not defined on } N = \{x \in A : |f(x)| = \infty, g(x) = 0 \lor |g(x)| = \infty, f(x) = 0\}
\end{cases}
\]

However, if \(f, g\) integrable then \(|f| < \infty\) and \(|g| < \infty\) a.e. in \(A\), in which case we say \(f + g, fg\) integrable and we denote \(\int_A (f + g) = \int_{A \setminus N} (f + g)\) and \(\int_A fg = \int_{A \setminus N} fg\).

Proposition 3.15. Let \(f, g : A \to \mathbb{R}\) be integrable. Then

1. \(\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g\) is integrable and \(\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g\).

2. \(\forall B_1, B_2 \subseteq A\) measurable disjoint, \(\int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f\).

3. \(f \leq g\) on \(A\) \(\Rightarrow \int_A f \leq \int_A g\). \(f = g\) a.e. on \(A\) \(\Rightarrow \int_A f = \int_A g\).

4. \(\int_A f \leq \int_A |f|\).

Theorem 3.16 (Dominated Convergence Theorem). Let \((f_n)_{n \in \mathbb{N}}\) be measurable functions on \(A\) such that

1. \(\exists f : A \to \mathbb{R}\) measurable such that \(\lim_{n \to \infty} f_n(x) = f(x)\) for a.e. \(x \in A\), and

2. \(\exists g : A \to \mathbb{R}\) integrable such that \(|f_n(x)| \leq g(x)\) for a.e. \(x \in A\) and \(\forall n \in \mathbb{N}\).

Then \(f_n\) and \(f\) are integrable and \(\lim_{n \to \infty} \int_A f_n = \int_A f\).
Corollary 3.17 (continuity of the integral). Let \( f \) be integrable over \( A \subseteq \mathbb{R}^d \). Then

1. If \((A_n)_{n \in \mathbb{N}}\) is a sequence of measurable subsets of \( A \) such that \( A_n \subseteq A_{n+1} \) then
   \[
   \int_{\bigcup_{n=1}^{\infty} A_n} f = \lim_{n \to \infty} \int_{A_n} f
   \]

2. If \((A_n)_{n \in \mathbb{N}}\) is a sequence of measurable subsets of \( A \) such that \( A_n \supseteq A_{n+1} \) then
   \[
   \int_{\bigcap_{n=1}^{\infty} A_n} f = \lim_{n \to \infty} \int_{A_n} f
   \]

4 Fubini and Tonelli’s Theorems

Definition 4.1. Let \( d_1, d_2 \in \mathbb{N} \) be such that \( d = d_1 + d_2 \). We denote \((x, y) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \). For every \( E \subseteq \mathbb{R}^d \), we denote \( E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in E\} \) and \( E_y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\} \).

\( \forall f : E \to \mathbb{R}, f_x : E_x \to \mathbb{R}, y \mapsto f(x, y) \) and \( f_y : E_y \to \mathbb{R}, x \mapsto f(x, y) \)

Remark. \( E_x \) and \( E_y \) are not necessarily measurable when \( E \) is measurable.

Remark. It is not always true that \( \int_A (\int_B f(x, y)dy)dx = \int_B (\int_A f(x, y)dx)dy \) even when the integrals are well-defined.

Theorem 4.1 (Fubini). Let \( f : \mathbb{R}^d \to \mathbb{R} \) be integrable. Then

1. For a.e. \( y \in \mathbb{R}^{d_2}, f_y \) is integrable on \( \mathbb{R}^{d_1} \),

2. \( y \mapsto \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^{d_1}} f(x, y)dx \) is integrable on \( \mathbb{R}^{d_2} \), and

3. \( \int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x, y)dx)dy = \int_{\mathbb{R}^d} f \).

Remark. The roles of \( x \) and \( y \) can be interchanged so that \( \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_1}} (\int_{\mathbb{R}^{d_2}} f(x, y)dy)dx \).

Theorem 4.2 (Tonelli). Let \( f \) be nonnegative measurable on \( \mathbb{R}^d \). Then

1. For a.e. \( y \in \mathbb{R}^{d_2}, f_y \) is measurable in \( \mathbb{R}^{d_1} \),

2. \( y \mapsto \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^{d_1}} f(x, y)dx \) is measurable in \( \mathbb{R}^{d_2} \), and

3. \( \int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f_y) = \int_{\mathbb{R}^d} f \).
Corollary 4.3. If $A \subseteq \mathbb{R}^d$ is measurable then for a.e. $y \in \mathbb{R}^d$, $A_y$ is measurable and moreover, $y \mapsto m(A_y)$ is measurable and $m(A) = \int_{\mathbb{R}^d} m(A_y) dy$.

Corollary 4.4 (Tonelli for $A \subseteq \mathbb{R}^d$). Let $f : A \rightarrow \mathbb{R}$ be nonnegative measurable. Then

1. For a.e. $y \in \mathbb{R}^d$, $f_y$ is measurable in $\mathbb{R}^d$,
2. $y \mapsto \int_{\mathbb{R}^d} f_y$ is measurable in $\mathbb{R}^1$, and
3. $\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f_y) dy = \int_{\mathbb{R}^d} f$.

Corollary 4.5 (Fubini for $A \subseteq \mathbb{R}^d$). Let $f : A \rightarrow \mathbb{R}$ be integrable over $A$. Then

1. For a.e. $y \in \mathbb{R}^d$, $f_y$ is integrable on $\mathbb{R}^d$,
2. $y \mapsto \int_{\mathbb{R}^d} f_y(x,y) dx$ is integrable on $\mathbb{R}^d$, and
3. $\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} f(x,y) dx) dy = \int_{\mathbb{R}^d} f$.

Lemma 4.6. $\forall E_1 \subseteq \mathbb{R}^d, E_2 \subseteq \mathbb{R}^d$,

$$m_*(E_1 \times E_2) \leq \begin{cases} m_*(E_1)m_*(E_2) & m_*(E_1) \neq 0 \land m_*(E_2) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4.7. Let $E_1 \subseteq \mathbb{R}^d_1$ and $E_2 \subseteq \mathbb{R}^d_2$ be measurable. Then $E_1 \times E_2$ is measurable and

$$m_*(E_1 \times E_2) = \begin{cases} m_*(E_1)m_*(E_2) & m_*(E_1) \neq 0 \land m_*(E_2) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 4.8. Let $E_1 \subseteq \mathbb{R}^d_1$ and $E_2 \subseteq \mathbb{R}^d_2$ be measurable and $f$ be a measurable function on $E_1$. Then $\tilde{f} : E_1 \times E_2 \rightarrow \overline{\mathbb{R}}, (x,y) \mapsto f(x)$ is measurable on $E_1 \times E_2$.

Proposition 4.9. Let $d_1 = d - 1$, $A \subseteq \mathbb{R}^{d_1}$ be measurable and $f : A \rightarrow [0, \infty]$. Then $f$ is measurable if and only if $E = \{(x,y) \in A \times \mathbb{R} : 0 \leq y \leq f(x)\}$ is measurable. Furthermore, if $f$ is measurable, then $m(E) = \int_A f(x) dx$.

Proposition 4.10. Let $f$ be measurable on $\mathbb{R}^d$. Then $g : \mathbb{R}^{2d} \rightarrow \overline{\mathbb{R}}, (x,y) \mapsto f(x-y)$ is measurable.

Remark. This is useful when defining convolution $f * g : x \mapsto \int_{\mathbb{R}^d} f(x-y)g(y)dy$. 

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5 Differentiation

**Theorem 5.1.** A monotone function \( f : [a, b] \to \mathbb{R} \) is differentiable almost everywhere in \((a, b)\). Furthermore, \( f' \) is integrable and

\[
\int_a^b f' \begin{cases} 
\leq f(b) - f(a) & f \text{ increasing} \\
g \geq f(b) - f(a) & f \text{ decreasing}
\end{cases}
\]

**Remark.** The Cantor-Lebesgue function is monotone, differentiable in \([0, 1]\), with \( \varphi' = 0 \) a.e. in \([0, 1]\) but \( \int_0^1 \varphi' = 0 < \varphi(1) - \varphi(0) = 1 \).

**Theorem 5.2.** Let \( F \) be a collection of bounded intervals in \([a, b] \subseteq \mathbb{R}\) of positive length. Then there exists a countable collection \( F' \subseteq F \) of disjoint intervals such that \( \bigcup_{I \in F} I \subseteq \bigcup_{I \in F'} 5I \), where \( 5I = \{x \in \mathbb{R} : x_I + \frac{1}{5}(x - x_I) \in I\} \) (\( x_I \) middle point of \( I \)).

**Remark.** It is possible to replace 5 by a number \( x > 3 \) but no less: consider \( F = \{[-1, 0], [0, 1]\} \).

**Proposition 5.3.** A monotone function \( f : [a, b] \to \mathbb{R} \) has at most countably many discontinuities.

**Functions of bounded variation**

**Definition 5.1.** Let \( f : [a, b] \to \mathbb{R} \) be a function. We call total variation of \( f \) on \([a, b]\) the number

\[
T_f(a, b) = \sup \left\{ \sum_{i=1}^k |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \cdots < x_k = b \right\}
\]

If \( T_f(a, b) < \infty \), then we say that \( f \) is of bounded variation on \([a, b]\).

**Remark.** Monotone and Lipschitz continuous functions are of bounded variation.

**Remark.**

\[
f(x) = \begin{cases} 
x \cos \left( \frac{1}{x} \right) & 0 < x \leq 1 \\
0 & x = 0
\end{cases}
\]

is not of bounded variation.
Theorem 5.4. A function \( f : [a, b] \rightarrow \mathbb{R} \) is of bounded variation if and only if it can be written as the difference between two increasing functions. In particular, if \( f \) is of bounded variation then \( f \) is differentiable a.e. and \( f' \) is integrable over \([a, b]\).

Absolutely continuous functions

Definition 5.2. We say that a function \( f : [a, b] \rightarrow \mathbb{R} \) is absolutely continuous on \([a, b]\) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for every finite collection of disjoint open bounded intervals \((a_k, b_k) \subseteq [a, b], 1 \leq k \leq n\), if \( \sum_{k=1}^{n} (b_k - a_k) < \delta \) then \( \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon \).

Remark. \( f \) absolutely continuous \( \Rightarrow \) \( f \) uniformly continuous by taking \( n = 1 \).

Remark. The Cantor-Lebesgue function \( \varphi \) is not absolutely continuous on \([0, 1]\).

Proposition 5.5. If \( f : [a, b] \rightarrow \mathbb{R} \) is Lipschitz continuous then \( f \) is absolutely continuous on \([a, b]\).

Theorem 5.6. If \( f : [a, b] \rightarrow \mathbb{R} \) is absolutely continuous on \([a, b]\) then \( f \) can be written as the difference between two increasing absolutely continuous functions. In particular, \( f \) is of bounded variation on \([a, b]\).

Theorem 5.7. Let \( f : [a, b] \rightarrow \mathbb{R} \).

1. If \( f \) is absolutely continuous on \([a, b]\) then

   \[
   \forall x \in [a, b] : \int_{[a, x]} f' = f(x) - f(a)
   \]

2. Conversely, for every integrable function \( g \) over \([a, b]\), the function \( x \mapsto \int_{a}^{x} g \) is absolutely continuous on \([a, b]\) with derivative equal to \( g \) a.e. in \([a, b]\).

Lemma 5.8. Let \( h \) be integrable over \([a, b]\). Then \( h = 0 \) a.e. in \([a, b]\) \( \iff \int_{a}^{b} h = 0 \) for all \( x \in (a, b) \).

Corollary 5.9. If \( f : [a, b] \rightarrow \mathbb{R} \) is monotone, then \( f \) is absolutely continuous in \([a, b]\) \( \iff \int_{a}^{b} f' = f(b) - f(a) \).

Corollary 5.10 (Lebesgue decomposition). Every function \( f : [a, b] \rightarrow \mathbb{R} \) of bounded variations can be written as \( f = f_{\text{abs}} + f_{\text{sing}} \), where \( f_{\text{abs}} = \int_{a}^{x} f' \) is absolutely continuous in \([a, b]\) and \( f_{\text{sing}} = f - f_{\text{abs}} \) is such that \( f'_{\text{sing}} = 0 \) a.e. in \([a, b]\).